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# math workshop
## for children

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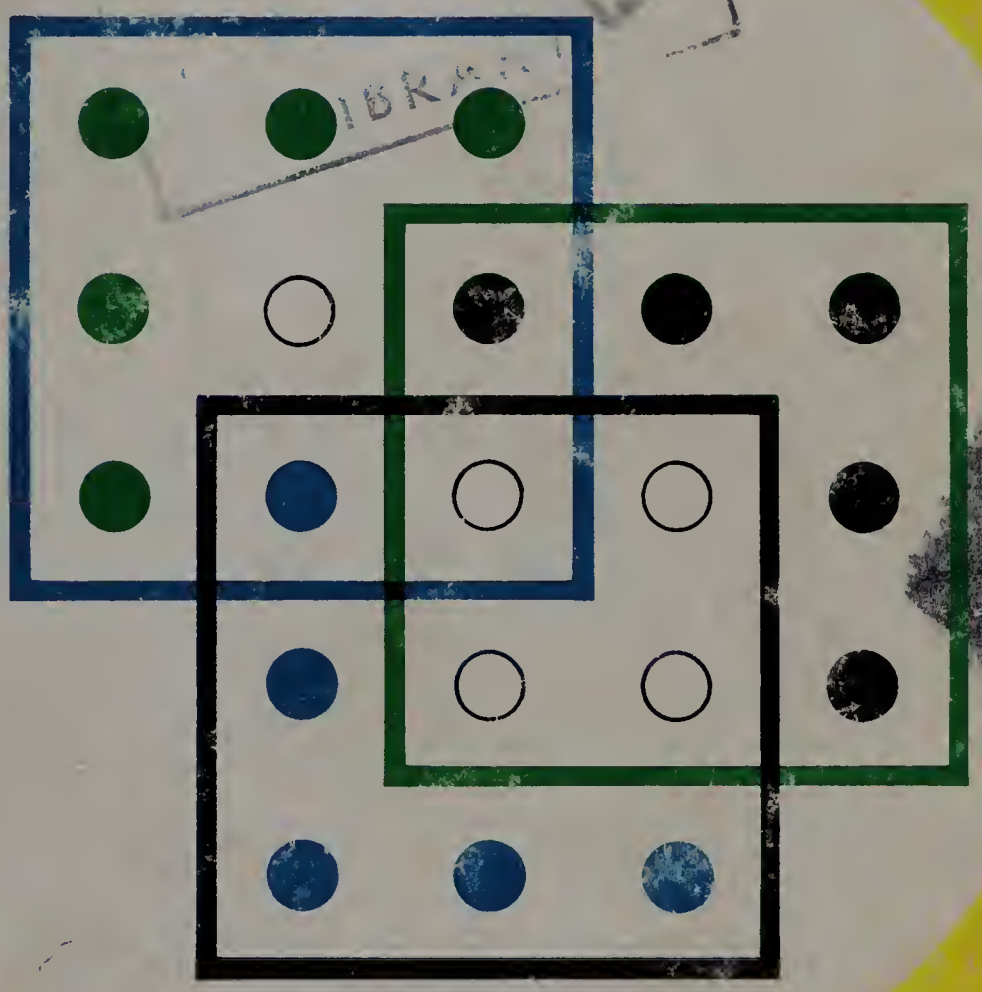
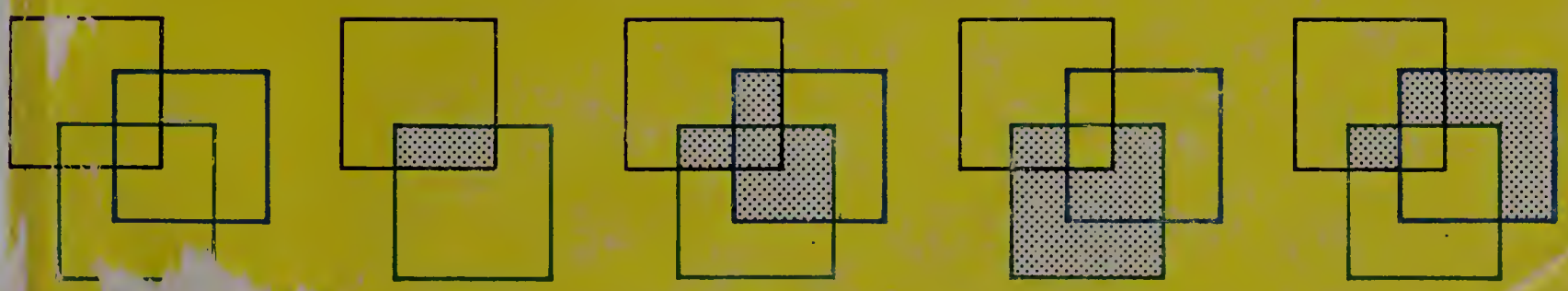
## general guide



ROBERT W. WIRTZ
MORTON BOTEL



# MATH WORKSHOP FOR CHILDREN

**Robert W. Wirtz**

Staff Research Associate, Addressograph-Multigraph Corporation and Mathematics Consultant, Bucks County Schools, Pennsylvania

**Morton Botel**

Assistant Superintendent and Reading Consultant, Bucks County Schools, Pennsylvania

**Advisor**

**W. W. Sawyer**

Professor of Mathematics, Wesleyan University, Connecticut

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# Introduction

The reputation of mathematics today resembles that of the automobile 50 years ago. Then cars were thought to be expensive and dangerous. It was not believed that anyone but a rich man would ever be able to own one, or anyone but a specially trained chauffeur able to drive one. In the same way mathematics is still believed to be for the exceptional person, for the aristocracy of intellect, for the few.

It is time that someone did for mathematics what Ford did for cars when he produced his model T.

Not only do industry and science need more high-powered mathematicians than the few thousand available at present, but it is becoming necessary for the entire population to have a basic understanding of mathematics and science, in order to cope with the growing complexity of life and the ever-accelerating pace of change.

The demand for more Ph.D.'s in mathematics might be met by concentrating students of marked mathematical ability in a few schools where, from their earliest years, they would be taught by specially trained teachers. But the education of the entire nation cannot be achieved by such means. We cannot provide courses for all the teachers in all the elementary schools. The teachers themselves are too busy to take such courses, or even to follow elaborate manuals of instruction. An improvement of mathematical education on the widest scale can only be achieved through new texts which must be attractive, self-explanatory and mathematically sound.

The Math Workshop for Children meets these requirements to a remarkable degree. The material is familiar. Teachers do not have to acquire a new language or take a lengthy course in higher mathematics. Here are the ordinary numbers of arithmetic as we have always known them. But interesting patterns are woven with these familiar threads. Stimulating questions are asked that bring out the originality and creative talents of children. Without high-sounding phrases or affectation of learnedness the students are led to develop a way of thinking and an interest in mathematical patterns that prepare them for more advanced work in later years.

The thinking of the mathematician is not essentially different from the thinking all of us do about our everyday affairs—except perhaps that there is an artistic and exciting element that is usually absent in our more mundane thinking. Most mathematicians, however, have been remarkably unsuccessful in communicating their thoughts to the general public.

The Math Workshop for Children brings out the delight and the simplicity that are part of real mathematics. It is the most valuable contribution yet made to the teaching of elementary mathematics in this country.

PROF. W. W. SAWYER
*Department of Mathematics*
Wesleyan University
Middletown, Connecticut



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# Contents


INTRODUCTION

MATERIALS OF THE MATH WORKSHOP FOR CHILDREN

PREFACE

**Part One**

FROM FRAGMENTS TO PATTERNS

I A Cross Number Puzzle

II Behind the Story

III Chain Reactions Problems

IV Following A Pattern

**Part Two**

THE UNIFYING IDEAS OF MATHEMATICS

I Number and Counting

II Equality and Inequality

III Numeration

IV The Basic Facts and Operations of Arithmetic

A. Addition

1. Whole Numbers
2. Fractions
3. Subtraction—the Inverse of Addition

B. Multiplication

1. Whole Numbers
2. Fractions
3. Division—the Inverse of Multiplication

a. Division Algorisms

b. Division of Fractions

V Tables and Number Pairs

VI Measurement

VII Problem Solving

A. Mathematical Headlines

B. Tables

C. Pure Mathematical Problems

VIII Geometry

A FINAL WORD

# The Materials of the Math Workshop for Children

## I. Basic Materials

A. GENERAL MANUAL TO THE MATH WORKSHOP FOR CHILDREN

This orientation booklet is to help the teacher understand the background, development, approach, and scope of the Math Workshop for Children.

B. INDIVIDUALIZED PUPILS' WORK TEXTS

Sets of 128 or more loose-leaf pages for each level.

| *Grade* | | *Designation* |
|---|---|---|
| 1 | — | A |
| 2 | — | B |
| 3 | — | C |
| 4 | — | D |
| 5 | — | E |
| 6 | — | F |

C. TEACHING AIDS

Teacher Editions for each grade.

Each page of each grade of the Work Texts is completed and annotated, and is accompanied by a detailed practical guide page.

## Supplementary Materials

A. Work-Forms.

Th ese forms help pupils explore mathematical ideas in greater depth.

B. The Mastery Tests of the 390 Basic Facts of Arithmetic. They are prepared on preprinted masters (Ditto or Multilith).

Six sets of tests of the 390 basic facts of arithmetic presented in several forms, a Progress Record Chart for each pupil to record his status and improvement, and two Check Sheets for him to use in marking his own answers. These are prepared on preprinted masters (Ditto or Multilith).

C. Manual for the Work-Forms and Mastery Tests

# Preface

Mathematics is a world of endless fascination. It is a world brimming with surpr
patterns, and enticing vistas. To see a pattern emerge from a conglomeration of f
to assemble seemingly disconnected items into a structure that has meaning—t
are some of the joys of mathematics. It is this sense of excitement in mathem
which pupils ought to be learning in elementary schools.

Giant steps are currently being taken to alter the traditional approach to teac
mathematics in order to get meaty mathematics into the elementary and secon
curricula. To this end, brilliant pioneering work has been done at the Universiti
Illinois, Maryland, Syracuse, Stanford, and elsewhere. Generous financial suppor
been given by the National Science Foundation, the Carnegie Corporation and u
the National Defense Education Act to support research projects and to sp
workshops and institutes. The National Council of Teachers of Mathematics i
24th Yearbook, *The Growth of Mathematical Ideas K-12*, has prepared a comprehe
statement of what the aim of mathematics instruction should be from Kinderg
to the 12th grade and beyond.

Fundamentally, the recommendations of the various groups which are strivi
improve the teaching of mathematics can be summarized as follows:

> Mathematics must be taught in a way which focuses on the fundamental stru
> of mathematics. Pupils must learn that arithmetic is not just a collection of fragm
> but rather is a mathematical system organized according to definite principles
> pupils to understand this concept, methods of instruction must be developed
> place the emphasis on mathematical problems that reveal these principles and re
> their use.

Educators who have been directly concerned with mathematics instruction
posed such questions as:

> At what age are children ready to study mathematics?
>
> How can we provide for the wide range of individual differences within any age g
>
> How can we help pupils discover ideas for themselves?
>
> How can we make learning meaningful?
>
> How can we motivate pupils so they will want to learn?

To date we have not had very effective answers to these questions.

Much of the reason for the ineffectiveness in teaching mathematics can be tra
a narrow concept of what mathematics is. If our sights are focused upon the

athematical position, i.e. that mathematics is best learned in terms of its structure, is possible to develop materials and approaches which provide properly for readiness, r discovery, for meaning, for individual differences, and for motivation.

A statement from the U.S. Office of Education sets forth clearly the need for a new pproach to mathematics to meet the demands of our dynamic, progressing society.

"Space missiles are only symbols of the great explosion of scientific knowledge of the 20th century. One of the most important factors contributing to this explosion is the revolutionary advance in both the development and the use of mathematics.

"Not only are new requirements being placed on mathematics in the fields of physics, chemistry and engineering, but in other fields mathematics is being put to new and even more astonishing uses.

"The biologist is applying mathematical theory to the study of inheritance; industry is using mathematics in scheduling production and distribution; the social scientist is using ideas from modern statistics; the psychologist is using mathematics of game theory. In fact, the logic of mathematical models shows promise as the basis for developing teaching machines for all areas of knowledge. The new uses of mathematics require less manipulation of formulas and equations but greater understanding of the structure of mathematics and mathematical systems. There is less emphasis on human computation that can be done by machines, and more emphasis on the construction of mathematical models and symbolic representation of ideas and relationships. Because of these new uses, mathematics is being firmly woven into the fabric of the national culture. The role of mathematics is not only to grind out answers to engineering problems, but to produce mathematical models (prototypes) that forecast the outcome of social trends and even the behaviorial changes of the group. Such important new uses and interpretations of mathematics require that students have a program with a greater depth than the classical program designed for 19th century education."

The Math Workshop for Children is a program that tries to reflect the thinking of p mathematicians and educators on what needs to be done to improve the teaching )elementary school mathematics. The program has been in the making for more than ır years. It was launched when its senior author tried to devise more imaginative ıterials to help pupils explore the wonderland of mathematics. He tried out these ıterials for several years with children in a Math Club that met in his home and in ıssrooms from grades one to six.

Encouraging responses from delighted pupils who were exposed to the early versions the material led to a collaboration with a curriculum director and author who had ıny years of experience in developing curriculum programs. Dr. W. W. Sawyer, ofessor of Mathematics at Wesleyan University, Middletown, Connecticut, became nsultant to the program. When materials had been prepared for all elementary grades,

a tryout was launched with over 4,000 pupils during 1960-1961 in the schools of B County, Pennsylvania, and in several other districts in Pennsylvania and New Je The 135 teachers and 15 supervisors of these pupils took an in-service course to them understand the new approaches to the teaching of mathematics.

The organization and content of the present edition of the Math Worksho Children (grades 1 to 6 inclusive) have been significantly influenced by the experi of the authors as they worked with pupils, teachers, and supervisors. Practical gestions made by teachers and supervisors in the tryout groups have been of great v

Eleven key features are incorporated in the program to provide excitement, inte and a sensitive educational environment.

**The basic content is aimed at the development of the unifying ideas of mathem** These unifying ideas include: (1) number and counting, (2) equality and inequa (3) numeration, (4) the basic facts of arithmetic and the operations of addition multiplication and their respective inverses, subtraction and division, (5) tables number pairs, (6) measurement, (7) problem solving, and (8) geometry.

**Each mathematical idea is taught in a spiral fashion.** Each of the unifying ide mathematics is introduced in the first grade. Each idea is frequently re-examine each succeeding grade. Although the child's mind is far from ready to master t ideas in all their generality, seeds of each can be planted early. These will grow by year in the right climate, into mature understanding.

**Four major provisions are made for individual differences in ability.** (a) Each at every grade level focuses on one or more of the unifying ideas of mathem Many pages are divided into three project areas. The first area presents the sim treatment of the idea. The area marked in green offers the pupil an opportuni explore the idea more fully, and the area in blue offers the bright student an opport to explore even further. Although no pupil need go beyond the first area to unders the idea, more capable pupils can be encouraged to advance their explorations. Teacher's Guide page for every pupil's page suggests additional activities for capable pupils as well as for slower pupils. (c) Supplementary material on Ditt Multilith masters is provided for pupils to experiment with. Since these are ope activities, pupils at every level can find success and challenge. (d) Six tests of th basic facts of arithmetic are provided with a Progress Record Chart and Check S for self-scoring and almost immediate reinforcement. By means of these tests, the can discover and work on the specific combinations that give him difficulty.

**Color is used functionally.** It is used to reveal patterns, to greatly condense the space, and to clarify instructions.

**Pictures have been prepared which suggest mathematical ideas.** For exampl relationship between addition and its inverse, subtraction, is suggested in doze illustrations which represent this idea. Moreover, pupils are encouraged to their own pictures.

**Numbers and geometric forms are recorded by pupils in tables, graphs and other ual displays.** Almost every page when completed reveals fundamental patterns in thematics that are interesting for the pupil to discover and explore. These built-in tterns serve as learning reinforcements, since a break in a pattern is immediately parent.

**Continuous emphasis is put on "discovery" and on encouraging mathematical cussions.** Mathematical discoveries do not result from questions that are easily swered by a number. They result from discussions that lead to inquiry. The design the text and guide pages encourages the teacher to ask: "What is going on here?" Why did that happen?" "Can you predict the next combination of numbers?" "What he pattern on this page?" "How many ways can you describe this pattern?" Trying answer such questions helps the pupil to expand his mathematical reasoning ability.

**The reading problem is minimized.** Pupils with reading problems often are blocked conventional mathematics programs by the overwhelming amount of reading volved. A minimum number of words is used in the Math Workshop for Children. a result, children who have poor reading ability can move ahead faster than normally uld be expected. Problem solving is not restricted to "doing" word problems.

**"Numo", our fantasy character, is a fascinating arithmetic explainer.** Throughout pages of the program Numo and his family and friends appear. Numo has a great rst for exploring mathematical ideas and for finding out what makes things tick. t he is a "real boy" too, getting into trouble, playing tricks and otherwise making nself delightful to watch, think about, and to draw.

**The instructional guides are functional and realistic.** This general manual presents over-all structure of the Math Workshop for Children in terms of specific activities. addition, it contains a guide page that corresponds to each pupil's page with step--step suggestions. In the Teacher's Guide, the text pages are completed in red. A sepae Guide is provided for use with the Supplementary Forms and the Mastery Tests.

**A loose-leaf format is used.** This flexible approach to organizing a sequence of pages s many advantages. First, the pupil can easily take a page out of the binder and work it. Second, the pupil can take an individual page home to explain his school work his family. Third, the pupil can easily add blank forms to the loose-leaf binder. nally the pupil prefers the loose-leaf format because to him it seems more adult.

In each grade, all basic ideas are reintroduced and reinforced before they are exnded to higher levels. Consequently, when children transfer into the Math Workshop Children from other programs, they will be confronted at first with nothing outside ir grasp.

Further, the mathematical situations at every grade level include all the content t other programs and achievement tests have assigned to that level. Therefore, ry child in the Math Workshop for Children will be presented with all the facts d processes that he will be expected to know if he transfers to some other program.

Finally, the Math Workshop for Children provides pupils with the tools to tac all mathematical and social arithmetic situations—including those in standardi achievement tests. In short, children can enter and leave this program without a set-backs.

This general manual of the Math Workshop for Children is organized into t sections, each serving a particular objective.

Part I, From Fragments to Patterns, was written to open the mind of the teache the "surprises and the patterns and enticing vistas of mathematics."

Read this section with a pencil in hand. Work through some of the exercises provid Make up some of your own examples. Only by actively participating will you see patterns, make the discoveries, and feel the movement of mathematics.

Part II, the Unifying Ideas of Mathematics, presents the scope and organizatio the Math Workshop for Children. This section explains the approach of the M Workshop for Children in teaching what Phillip S. Jones[1] calls the " . . . basic ma matical understandings which should be continually developed and extended throu out the entire mathematics curriculum, grades K-12 and beyond."

Parts I and II include many examples of the kinds of experiences pupils get in Math Workshop for Children materials to help them understand and enjoy mathema

ROBERT W. WIRT

MORTON BOTEL

[1] The National Council of Teachers of Mathematics. THE GROWTH OF MATHEMAT IDEAS GRADES K-12. (The 24th Yearbook) Washington, D. C. 1959

# Part One
# From Fragments to Patterns

## A CROSS NUMBER PUZZLE

A drill exercise in a traditional first or ond grade arithmetic program might k like this:

Add the following:

| | | | | | |
|---|---|---|---|---|---|
| 2 | 1 | 2 | 2 | 3 | 1 |
| 1 | 4 | 3 | 4 | 1 | 2 |
| 3 | 5 | 5 | 6 | | |
| 4 | 1 | 3 | 3 | 4 | 4 |
| 2 | 3 | 4 | 2 | 1 | 3 |

This is an example of practice for 12 of e 100 basic addition facts.

Some programs also use the form:

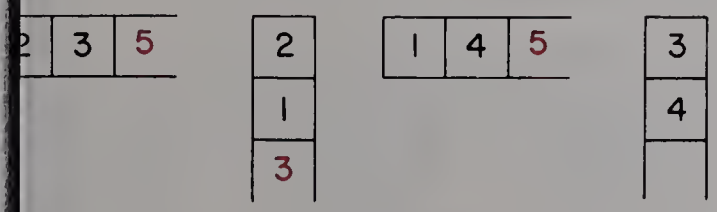
2+1=___ 1+4=___

2+3=___ etc.

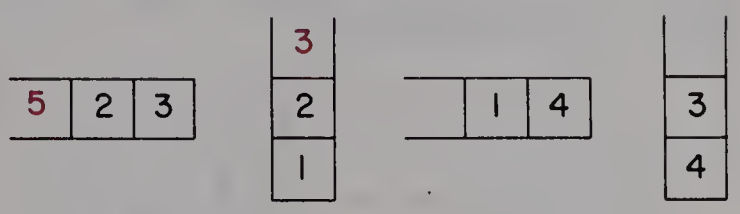
ually the two forms are mixed—so ldren will not become dependent upon particular form.

But why stop with these two forms?

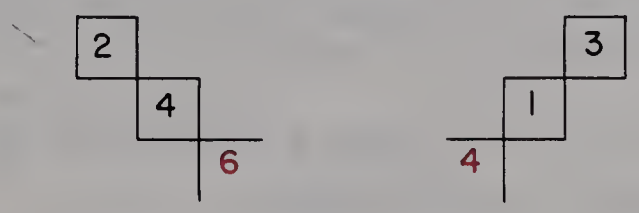
Add the following and place the sums the incomplete boxes:



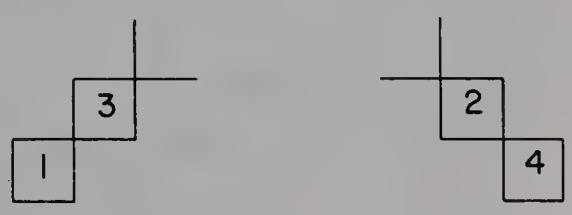

The next arrangement adds another ist—a backward and an upside down rm.



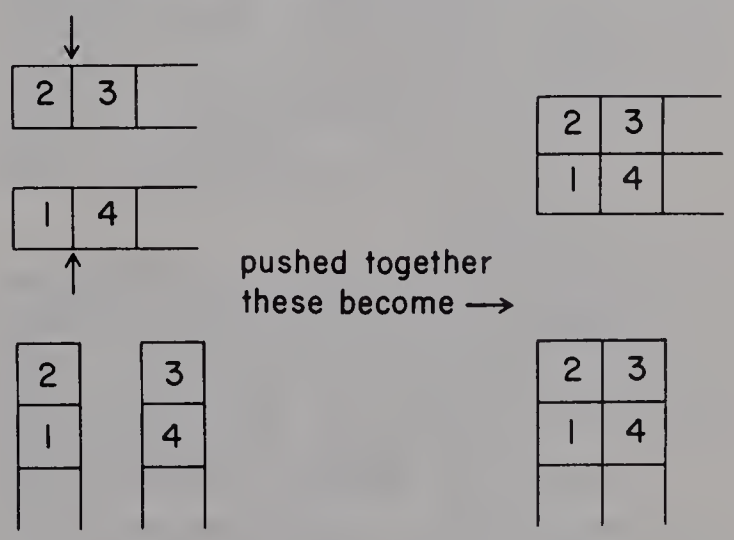

But there is no need to become a slave to vertical and horizontal forms. How about a diagonal-down approach?



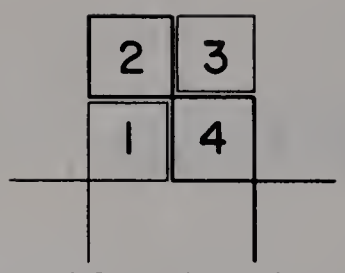

And—a diagonal-up approach?



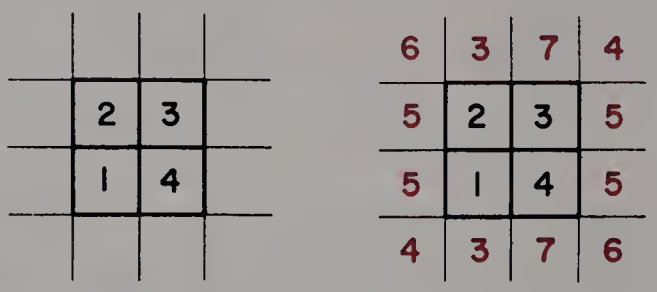

Notice that these examples include each of the original 12 basic facts. Here they are united into a more compact form.




In another maneuver, still other forms can be united.




And finally, with a few lines added, all twelve forms are compressed into a nice, neat structure. Completed by the children, it looks like this:




Here are those same 12 addition facts presented in a unique fashion that adds some fun to the activity.

In fact, on looking inquiringly at the results, it is evident that:

1. Those 12 addition facts mentioned earlier have now been compressed into a fraction of the space they formerly occupied.
2. The pupils have been stimulated with a bit of curiosity, a little humor; and, with the final consolidation, a feeling of having gotten somewhere.
3. The completed form is a compelling demonstration that "order" doesn't make any difference in determining the sum of two numbers: $2+3=5$ and $3+2=5$ (the commutative law of addition).
4. There is still more to be learned from this structure. It invites lively, interesting class discussion. The teacher can get the discussion started by asking, "Do you see anything in this structure worth talking about?"

The wise teacher, at this point, is an inconspicuous moderator as the chil explore the numerous symmetries. opposite corners are alike. The top bottom middle pairs are alike—"3" "7". The side middle pairs are ali "5" and "5." The top and bottom d onals, as pairs, are alike—"6" and

And the sum of each of these pai "10." Does this always happen?

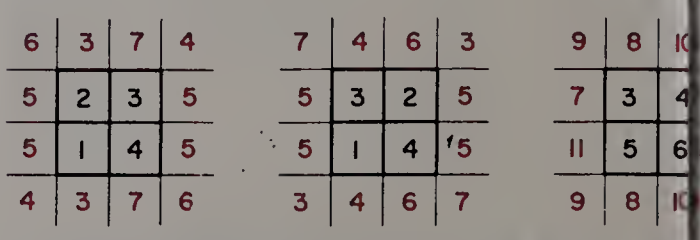
About this time a front runner blurt out that the sum of the four num in the central boxes is also 10—"1+ +4=10."

Perhaps someone will notice that, ta all together, those pairs, "6, 4"—"5, and "3, 7" all come from the serie numbers "3, 4, 5, 6, 7."

The conclusion is inescapable tha this particular example of compressin basic facts into a single form, the f result presents many patterns. Th patterns involve more than the orig 12 basic facts of addition.

How much of this example was tric How much depends on the choice particular group of numbers—"1, 2 and 4," in this case? Does the orde the numbers make any difference?

These questions lead to experime with (1) a change in arrangement, (2) a change in the central four bo For example:




Many similarities are obvious. A pattern was created when the numb

ere changed—all four corners are the ame. This time the "pairs" all yield 18. n out-in-fronter will discover that "3+4 -5+6" also "equals 18."

The class is now ready for individual ctivity. Pupils want to try their own ersions. Some want to try some bigger umbers. Others may wonder what hap- ens if "zero" is included in one or more f the center boxes. Do the numbers have o be in sequence? What must be true of he center boxes to make all four corners like? Can they discover new patterns; an they find examples that are without attern?

Now the teacher is free to work with ndividual children, and the children are ree to work at a level of difficulty each ets for himself. And the more talented ill see a new world opening—the open- nd search for more patterns in the struc- ure of arithmetic.

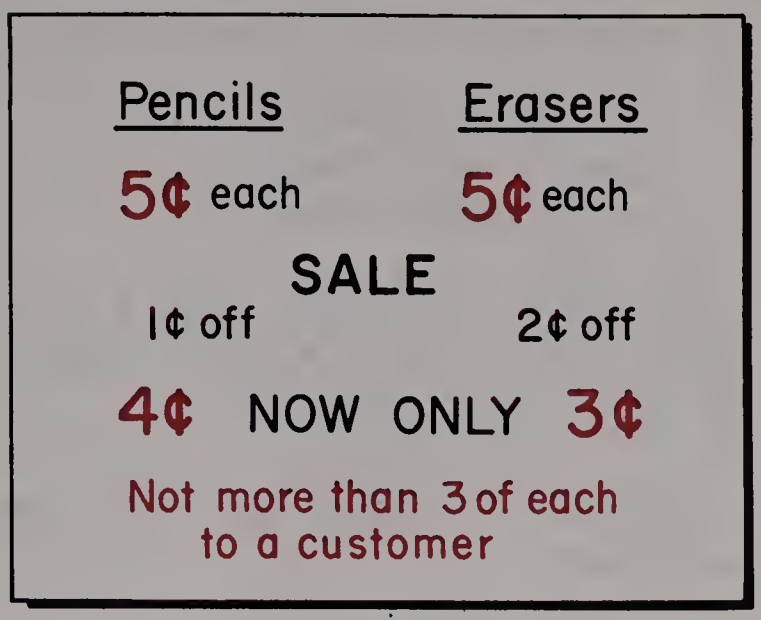
## . BEHIND THE STORY

Here are some "story problems" from standard text:

1. John had 9 marbles and lost 5. How many did he have left? (4 marbles)
2. Mary bought a pencil for 4¢ and an eraser for 3¢. How much did she spend? (7¢, of course)
3. Mr. Brown had 13 cows . . .

Wait a minute. Go back to problem 2. nce in a while it is interesting to con- struct the story behind the story problem.

Actually, when Mary went to buy her pencil and eraser, she had a surprise in store for her. Here is the sign she saw at the counter:




So, Mary was able to save 3¢ of the 10¢ she was prepared to spend. Perhaps she wondered whether she might buy two pencils and one eraser. No, not enough money. Well then, how about one pencil and two erasers?

If Mary had plenty of time, she might wonder about that "Not more than 3 of each to a customer." What would the largest possible purchase cost? Three pencils (4¢ each) would be 12¢ and 3 erasers (3¢ each) would be 9¢. And, that would be 21¢.

Of course, the smallest purchase would be one 3¢ eraser.

But how many different purchases could be made at this pencil-eraser counter with the limitation that had been put on this sale?

We already know the largest, and the smallest. How many are in between? Is there an unlimited number of combinations? Probably not, since we already know the largest—only 21¢.

How should this problem be tackled? There are many ways, of course. The most obvious way is to start out by listing

as many combinations as come to mind.

*1* pencil and *1* eraser
*2* pencils and *2* erasers
*3* pencils and *3* erasers

That little pattern got off to a good start, but it ran out in a hurry—stopped by the 3-to-a-customer limit.

Perhaps the customer might not need any erasers.

*1* pencil and *no* erasers
*2* pencils and *no* erasers
*3* pencils and *no* erasers

Stopped again. But it could be the other way around—if the customer had plenty of pencils.

*0* pencils and *1* eraser
*0* pencils and *2* erasers
*0* pencils and *3* erasers

Are there any possibilities that haven't been listed? A pattern seemed to get started without any effort—but each time, it ran into a roadblock. Each of the three patterns resulted in three possible purchases. Nine so far!

How about:

*2* pencils and *3* erasers?

And turn it around:

*3* pencils and *2* erasers?

Also:

*1* pencil and *3* erasers?

And turn that around:

*3* pencils and *1* eraser?

Combinations are getting harder to find. But the possibilities are not exhausted. Actually, there are two more than the 13 already discovered (excludi "no pencils and no erasers" because th is not a "purchase"). The missing co binations will appear quickly if you sta over and get a larger pattern going.

For example:

*3* pencils and *0* erasers
*3* pencils and *1* eraser
*3* pencils and *2* erasers
*3* pencils and *3* erasers

The next step is obvious.

*2* pencils and *0* erasers
*2* pencils and *1* eraser
*2* pencils and *2* erasers
*2* pencils and *3* erasers

And, keeping this pattern going:

*1* pencil and *0* erasers
*1* pencil and *1* eraser
*1* pencil and *2* erasers
*1* pencil and *3* erasers

And lastly:

*0* pencils and *0* erasers
*0* pencils and *1* eraser
*0* pencils and *2* erasers
*0* pencils and *3* erasers

But, the first combination in this la group must be crossed out because it isn a "purchase."

Now, all the possibilities have bee listed, keeping in mind the limitations "3 to a customer" and "none." How man combinations in all? . . . 4+4+4+3=1 . . . fifteen possible purchases.

How much would each of these pu chases amount to? The largest would b 21¢. What purchase would be next largest Could there be a purchase of 20¢?

Mary finally spent her whole dim

ying one pencil and two erasers—
-3+3=10. Could she have found
other combination that would total
¢?

Mary's friend, Alice, spent 8¢. What
she buy? Can you be sure that it was
pencils?

Bill heard about the sale. When he
ve the clerk a nickel and a dime, he
a penny change. What did Bill buy?

The busy clerk behind the counter
eded to know the price of each one of
se 15 possible purchases. Since he was
industrious and efficient clerk, he had
de a work-saver chart that looked
e this:

No. of Erasers at 3¢

| No. of Pencils at 4¢ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | X | | | |
| 1 | | | 10¢ | |
| 2 | | | | |
| 3 | | | | |

He had filled in all the prices, and when
ry asked for 2 erasers and 1 pencil, he
ked in the column under "2 erasers"
in the row next to "1 pencil." Where
y cross was written the price—"10¢."

Of course, everyone can find out the
ces that the clerk had written in his
ork-saver" and "mistake-eliminator"
rt. (The reader might like to run
ough the simple exercise himself. Look
short cuts as the pattern emerges.
er you have completed the chart, the
tern will be most obvious.)

At this point, children can take over on their own to fill in the chart. Some will need more help than others. As some pupils complete the chart, ask them to make up a chart the clerk might make after the sale—when the prices were again 5¢. The first pupils to complete both might be asked to put the "sale price" chart and the "regular price" chart on the board. They would look essentially like this with the obvious patterns indicated:

Erasers at 3¢

| Pencils at 4¢ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | X | 3 | 6 | 9 |
| 1 | 4 | 7 | 10 | 13 |
| 2 | 8 | 11 | 14 | 17 |
| 3 | 12 | 15 | 18 | 21 |

Erasers at 5¢

| Pencils at 5¢ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | X | 5 | 10 | 15 |
| 1 | 5 | 10 | 15 | 20 |
| 2 | 10 | 15 | 20 | 25 |
| 3 | 15 | 20 | 25 | 30 |

On examining the "sale price" chart, you will notice that the total price of each of the 15 possible purchases is unique—none is repeated. This fact permits the chart to be used to find the exact nature of any purchase if the total amount is known.

It is possible to answer conclusively the questions: "What did Alice buy that cost 8¢," and "What did Bill buy that cost 14¢?"

Perhaps some children would like to tackle the "why" of that diagonal pattern in the sale price chart. It would help to find the "why" of the similar but more obvious diagonal patterns in the "regular price" chart.

All this suggests an outline, at least, of a possible story behind the story-problem:

"Mary bought a pencil for 4¢ and an eraser for 3¢. How much did she spend?"

## III. CHAIN-REACTION PROBLEMS

All children need to tackle "hard-work" problems once in a while. A very significant aspect of these back-breakers is that they often are dramatic examples of how costly a careless error can be. One little slip can lead to a wrong result.

For example: John got the following wrong answer because of a mistake in adding the hundred dollar column.

| |
|---|
| $ 235.16 |
| 85.26 |
| 301.00 |
| 625.75 |
| 823.44 |
| $3,070.61 |

John committed a $1,000 error. His answer should have been $2,070.61. This is a good example of how 22 correct additions were thrown away in one instant of carelessness. Perhaps the worst of it all was that John was robbed of the feeling of confidence that comes from performing flawlessly a long chain of little steps.

It takes practice to be able to complete the back-breaker above quickly and accurately. This practice can be provided more interesting surroundings—surroundings which are also in the foothills of mathematical mountains. For example, there are "chain-reaction" problems.

The "rule" in the following chain-reaction problem is that each successive term or number is the sum of its two closest predecessors: 2, 3 will be followed by 5 (because 2+3=5) and 2, 3, 5 will be followed by 8 (because 3+5=8). So that the pupil will know if he is right or wrong, some check points are provided along the way.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| a. | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 |
| b. | 2 | 2 | | | | 16 | | | | 110 |
| c. | 3 | 3 | | | | 24 | | | | 165 |
| d. | | | | | | | | | | |

Slow pupils can stop at the first check point in each series—at "8", "16" and "24."

In a certain sense, "c" is the sum of "a" and "b." Each term in "c" is the sum of the two numbers listed above it. (3 is the sum of 2 and 1, 24 is the sum of 8 and 16, etc.)

Now fill in the terms of "d" by adding the three numbers listed above it. Does the series thus generated obey the rule? Is each term the sum of its two closest predecessors?

Another group of these chain reactions follows:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| e. | 1 | 2 | 3 | 5 | | 13 | | | | 55 |
| f. | 1 | 3 | | | | 18 | | | | 76 |
| g. | 1 | 4 | | | | 23 | | | | 97 |
| h. | 1 | 5 | | | | 28 | | | | 118 |
| i. | | | | | | | | | | |

Again, in row "i" put the sum of the numbers listed above each. Does this new

ries obey the rule?

Now start a series from the other end
d work backwards.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| j | 6 | 3 | 9 | 12 | 21 | 33 | 54 | 87 | | |
| k. | | | | | 12 | 19 | 31 | 50 | | |
| m. | | | | | | | | | | |

Since the rule is still in force, the term
eceding "54" can be arrived at by con-
lering "87." The term in question must
one which makes—___+54=87 into
completed and true statement. It must
"33."

Series "j" and "k" are so arranged that
ey can lead to "m" either by addition
subtraction—and in both cases the re-
lting series will obey the rule.

Using this rule, a series can be made to
ow in either direction indefinitely, as
on as any two consecutive terms are
ven.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 2 | 7 | 9 | 16 | 25 | | | | | | 455 | 733 |
| 4 | 8 | 12 | 20 | 32 | 52 | 84 | | | | | 932 | 1508 |

Or, the terms given might be separated
a missing term. And, of course, such
ries can be made to grow using fractions
decimals. (Further, with negative num-
rs, they grow indefinitely in *either*
rection.)

A slight change in the "rule," and a
fferent sort of chain reaction is created.
ppose it is decided that each term is the
oduct of its two predecessors.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| p. | 1 | 2 | 2 | 4 | | 32 | 256 | | | |
| q. | 2 | 3 | 6 | 18 | | 1872 | | | | |
| r. | 2 | 6 | | | | | | | | |

What happens if another series, "r," is
ken which is composed of the product
the two numbers listed above? Does it
ey the new rule? Yes, it does, and it is
quickly evident that this "product" rule leads to very rapid growth. The 7th term of "q" amounts to 108×1872.

Try another "rule." This time each term will be twice its predecessor.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| s. | 1 | 2 | 4 | 8 | | 32 | | | | |
| t. | 3 | 6 | 12 | | | 96 | | | | |
| u. | 5 | | | | | 160 | | | | |
| v. | 9 | 18 | | | | | | | | |

Find what rule "v" will follow if it is created by adding the three numbers listed above its position.

This rule (doubling the preceding term) makes it possible to start with any term and watch the growth on the ascending side as the terms are multiplied by 2 and on the descending side as they are divided by 2.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| w. | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 | 2 | 4 | 8 | 16 | | | |
| x. | $\frac{7}{8}$ | $1\frac{3}{4}$ | $3\frac{1}{2}$ | 7 | 14 | | | | | |

To those familiar with fractions, it is obvious that such series can grow indefinitely on the descending as well as ascending side.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| a. | $\frac{5}{16}$ | $\frac{5}{8}$ | $\frac{5}{4}$ | $\frac{5}{2}$ | 5 | 10 | 15 | | | |
| b. | $\frac{3}{8}$ | $\frac{3}{4}$ | $\frac{3}{2}$ | 3 | | | | | | |
| c. | | | | | | | | | | |

And if you generate "c" as sum of the two numbers listed above each term, the expected result emerges—the series is following the rule.

Of course, the "rule" can be changed so that each succeeding term is (a) three times, or (b) four times, or (c) two and one-half times its predecessor. Using the rule "½ times its predecessor" leads to an interesting development.

128 64 32 16 8 4 2 1 $\frac{1}{2}$ $\frac{1}{4}$

It brings us right back to the first series, using the rule "multiply by 2," only the chain has been turned end to end —it has been "inverted." Clearly then, dividing by 2 and multiplying by ½ lead to identical results.

Another kind of chain reaction gets under way with another rule and a new form.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 4 | 9 | 16 | ___ | ___ | 49 |
| 1 | 3 | 5 | ___ | ___ | ___ | 13 | |
| 2 | 2 | ___ | ___ | ___ | 2 | | |

The rule has two parts—(a) all terms in the bottom row are the same number, and (b) all others are the difference of two numbers—the one listed above left and above right (except the top row). So, the first term in the second row of the above example is 1 (1−0). The second term in that row is 3 (4−1). The second term in the bottom row is "2" because of the rule—that all terms in the bottom row are alike.

Complete the first example. The first row emerges as a familiar set of numbers —sometimes called the "square numbers." The second row is even more familiar— the "odd numbers." Again, terms at the end of each row act as "check points"— immediate rewards for flawless work or warning signs for imperfect computation.

"Chain-reactions" are a demanding exercise. One little careless error, and the whole endeavor gets into a mess. The faster pupil will quickly see that he can invent his own rules and build his own structures. (He will not realize it, but he will be exploring Fibonacci series, finite differences and algebraic equations. Don't be surprised if he spends hours with t kaleidoscopic patterns in this new land

The excitement of getting the chain r actions going starts off from one or tw terms and a simple rule. It involves almo any dimension of drill required, and th whole business takes place in a deepl satisfying mathematical environment.

## IV. FOLLOWING A PATTERN

Sometimes, tables are constructed ac cording to certain arbitrary rules. Ofte these tables reveal interesting and signif icant patterns as they grow.

$(1+1)\times(1-1)=1\times1-\square$
$(2+1)\times(2-1)=2\times2-\square$
$(3+1)\times(3-1)=3\times3-\square$
$(4+1)\times(4-1)=4\times4-\square$
$(5+1)\times(5-1)=5\times5-\square$

Surprising! So skip down the pattern to test its durability.

$(10+1)\times(10-1)=10\times10-\square$
$(11+1)\times(11-1)=11\times11-\square$
$(100+1)\times(100-1)=100\times100-\square$

Looks good, so build another table using "2" in place of the recurring "1."

$(2+2)\times(2-2)=2\times2-\square$
$(3+2)\times(3-2)=3\times3-\square$
$(4+2)\times(4-2)=4\times4-\square$
$(5+2)\times(5-2)=5\times5-\square$

And jump ahead for random testing.

$(10+2)\times(10-2)=10\times10-\square$
$(12+2)\times(12-2)=12\times12-\square$
$(25+2)\times(25-2)=25\times25-\square$

Something certainly is going on. Try substituting "3" for the recurrent "2."

$(3+3)\times(3-3)=3\times3-\square$
$(4+3)\times(4-3)=4\times4-\square$
$(5+3)\times(5-3)=5\times5-\square$
$(6+3)\times(6-3)=6\times6-\square$
$(13+3)\times(13-3)=13\times13-\square$

If a few pupils don't see the whole process that's going on here, you can make it stand out with the following variations:

$(5+1)\times(5-1)=5\times5-\square$
$(5+2)\times(5-2)=5\times5-\square$
$(5+3)\times(5-3)=5\times5-\square$
$(5+4)\times(5-4)=5\times5-\square$
$(5+5)\times(5-5)=5\times5-\square$

and

$(7+1)\times(7-1)=7\times7-\square$
$(7+2)\times(7-2)=7\times7-\square$
$(7+3)\times(7-3)=7\times7-\square$
$(7+4)\times(7-4)=7\times7-\square$
$(7+5)\times(7-5)=7\times7-\square$

and so on—until everyone has got it.

Now put it to work and test it at the same time.

$\times19$ can be written $(20+1)\times(20-1)=$
$20\times20-1=400-1=399$

$\times18$ can be written $(20+2)\times(20-2)=$
$20\times20-4=400-4=396$

$\times17$ can be written $(20+3)\times(20-3)=$
$20\times20-9=400-9-391$

$16\times14$ can be written $(15+1)\times(15-1)=$
$15\times15-1=225-1=224$

$17\times13$ can be written $(15+2)\times(15-2)=$
$15\times15-4=225-4=221$

$51\times49$ can be written $(50+1)\times(50-1)=$
$50\times50-1=2500-1=2499$

$102\times98$ can be written $(100+2)\times(100-2)=100\times100-4=10,000-4=9996$

The following examples illustrate a few more of the many variations of this technique that are possible:

$(1-1)\times1=(1\times1)-1$
$(2-1)\times2=(2\times2)-2$
$(3-1)\times3=(3\times3)-3$
$(4-1)\times4=(4\times4)-4$

Skip and test. . . .

$(10-1)\times10=(10\times10)-\square$

$(1+2)\times2=(1\times2)+\square$
$(2+2)\times2=(2\times2)+\square$
$(3+2)\times2=(3\times2)+\square$
$(4+2)\times2=(4\times2)+\square$

Skip and test. . . .

$(10+2)\times2=(10\times2)+\square$

$(1\times1)+(1\times2)=(1+2)\times1$
$(2\times2)+(2\times2)=(2+2)\times2$
$(3\times3)+(3\times2)=(3+2)\times3$
$(4\times4)+(4\times2)=(4+2)\times4$

Skip and test. . . .

$(10\times10)+(10\times2)=(10+2)\times\square$

There are similar table extensions that require more study to find a rule. How is it possible to find a solution without actually going through the indicated manipulations?

$(1\times2)-(0\times1)=2$
$(2\times3)-(1\times2)=4$
$(3\times4)-(2\times3)=6$
$(4\times5)-(3\times4)=8$

(One rule for this set might be: the solution must be twice the first number on the left.)

Skip to test the rule. . . .

$(8\times9)-(7\times8)=\square$

(By rule, the solution should be $8\times2$.)

$(1\times3)-(0\times2)=3$
$(2\times4)-(1\times3)=5$
$(3\times5)-(2\times4)=7$
$(4\times6)-(3\times5)=9$

(The rule for this set might be: one more than twice the first number; or, one less than twice the number prior to the equal sign.)

Test. . . .

$(9\times11)-(8\times10)=\square$

$(3\times2)-(1\times0)=\square$
$(4\times3)-(2\times1)=\square$
$(5\times4)-(3\times2)=\square$
$(6\times5)-(4\times3)=\square$

(Skip and test any rule that seems to the pattern.)

The foregoing is an excellent exercise fix the basic facts of arithmetic mo firmly in the memory. It has a built- characteristic that raises a signal at a most every step—does the pattern hol Often the results can be used as tim saving formulae, such as:

$19\times21=20\times20-1=399$*

* Of course, the exercise out of which this shor cut grew anticipates the generalizations that ( for every $x$, $(x+1)\times(x-1)=(x\times x)-1$ an (2) the more general conclusion that for every and every $y$, $(x+y)\times(x-y)=x^2-y^2$.

# Part Two
# The Unifying Ideas of Mathematics

The day arithmetic was fragmentized d given back in pieces, it was deprived its real content. For generations, text-oks have offered children a daily diet these fragments. Few authors have t these facts back together into the autiful structure that is mathematics. ildren have been denied the chance to ce the interlacing patterns that are the *rp and woof* of arithmetic.

The Math Workshop for Children has veloped a program that, at every grade el, puts the pieces of this "purest of e sciences" back together for young nds to contemplate. Only the complete ucture of mathematics can challenge eager mind to go on searching for the tterns that lie everywhere and lead off interesting directions.

In this section, each of the unifying as of mathematics is discussed in terms the specific techniques the Math Work-op for Children uses in guiding and veloping pupil understanding and dis-very.

## NUMBER AND COUNTING

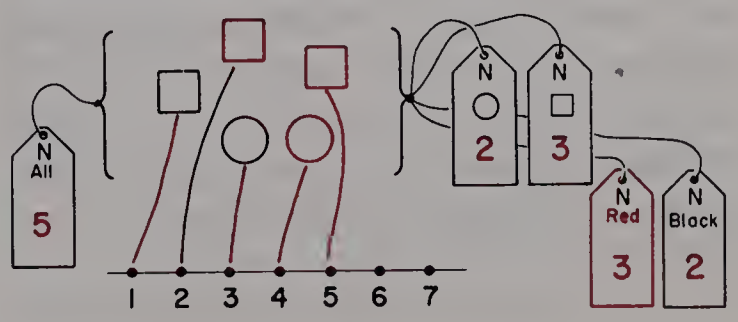
Kindergarten and first grade teachers ve long realized that "number" and ounting" are abstract ideas to be veloped patiently and carefully. They ow that most children must see many lections of three things before they se in all these situations a common perty we call "threeness." And they ow that when most children rattle off "one, two, three, four, etc." it is often no more than the recitation of nonsense sounds.

Good teachers know that pupils need to handle and arrange manipulative materials if they are to understand the ideas of number and counting. Such materials include: bottle tops, counters, buttons, Cuisenaire rods, and Stern structural materials.

As children experiment with these sensory aids, the teacher can guide them, encourage them to make patterns, talk about patterns and point to simple relationships among the different sizes, colors, and shapes. There will be many opportunities for pupils to talk about "none," "one," "two" and other numbers until finally these words will be fully understood.

As soon as pupils are ready, the Math Workshop for Children moves them on to making records of these counting experiences. Care is taken to deal with integrated pictures and activities rather than with isolated situations which are completed and forgotten as soon as the pupils have found the "correct answers."

Here is a typical example from the first-grade program.

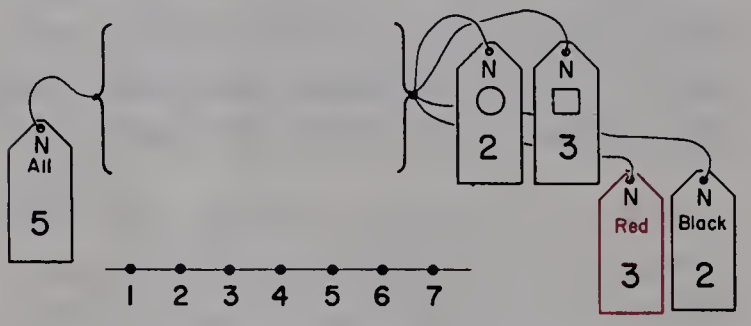


This is no casually thrown together picture of something to count. Each part is utterly simple, but each is tied to a larger experience.

Pupils begin by drawing lines, one from each element within the brackets to a mark on the number line—1, 2, 3, 4, 5. They can record this observation in the tag on the left—all together—by tally marks or with the numeral 5.

Other tags indicate that they are to record certain information concerning the number of squares and circles and the number of red shapes and black shapes. Again, tally marks or numerals are used to fill out the labels.

This collection has been counted in four different ways and the results recorded. In fact, all the information is present which one would need to reconstruct the picture if it were erased (which is the form we sometimes use to state the problem).

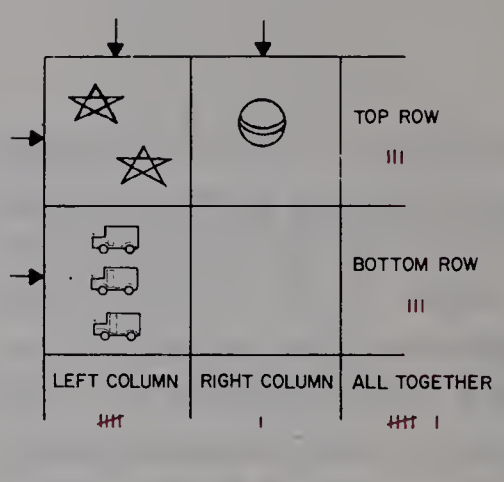



Further, the picture has been used in two ways that plant the idea of addition.

(5=3+2 and 5=2+3)

All of this has been accomplished without taking the focus away from this single example. There has been quite a discussion about the situation—enough to seed the first notions of more mathematical ideas to come. The business of record keeping has been initiated. In every aspect of the approach, fundamentally sound mathematics has been employed.

Cross number puzzles, with all their rich mathematical settings are introduc first on the elementary level of counting and records are kept. "Cross numb puzzle" is the name for such a device that shown below.*




* Over thirteen variations of the cross numb puzzle are used in the Math Workshop f Children.

Pupils count the objects in the top rov and put tally marks for each at the sid Or, they may write "3." They do th same with the bottom row. Then the count the column on the left, and th column on the right. In the lower right hand corner, they record the total of a objects.

Finally, the whole situation is recorde in the small diagram, again using tallie or numerals.

The first Work-Text does not hurr through counting. Instead, it provide many different kinds of experiences i many forms as different as the two pre sented above. In each, the setting is de signed to anticipate and to suggest th new ideas that are to follow—the shor cuts to counting called addition and mul tiplication, and their opposites, subtrac tion and division.

## II. EQUALITY AND INEQUALITY

Once they have a good understanding of number and counting, pupils are ready

another elementary mathematical con-
t—equality and its opposite, inequality.

One meaning of the word *is* bears a
se resemblance to the idea of mathe-
tical equality—abbreviated as "=."
ewise, "is not" suggests the mathe-
tical "not equal"—abbreviated as
."

*Mr. John Brown* is *Principal of our*
*ool.*

On each side of the word *is* occurs a
ne for a person, and both names desig-
e the same person. The sentence can
reversed to read "*The Principal of our*
*ool* is *Mr. John Brown.*" No other per-
's name can replace "Mr. Brown"
hout making the statement untrue.

Mr. Brown *is* Principal of our school.

Mr. Jones *is not* Principal of our school.

When the pupils pass beyond primitive
nting and begin to take short cuts
h as addition, the teacher will explain
t "3+2" *is* just "another name for 5."
h expressions name the same number.
the other hand 3+1 is not another
ne for 5.

$$3+2=5 \text{ but } 3+1\neq 5$$

Complete understanding of mathemati-
equality as the relation between two
nes for the same number is required
understanding the basic facts of
hmetic—and most of the mathematics
ollow.

Concepts of number and counting and
ality may take long to develop, but
must be understood before pupils
move on to addition and multipli-
on.

## III. NUMERATION

The fact that it is possible to name every number with only ten digits and positional values is one of the most fundamental and far reaching ideas in mathematics. The Math Workshop for Children provides early experiences which lead pupils to the discovery of the positional idea in our decimal system of numeration.

Right from the start, pupils work with a number line that has labels beyond 9. It has a section that looks like this:

They hear talk about "nine, ten, eleven, twelve." There is no barrier between nine and ten. They have no feeling of anything new and they move back and forth freely along this part of the number line. Something really important is there all right, but keep the secret for a while.

They are learning names for numbers—and "ten comes after nine," unless, of course, they are "counting backward" from eleven to ten to nine to eight. Then it comes before nine.

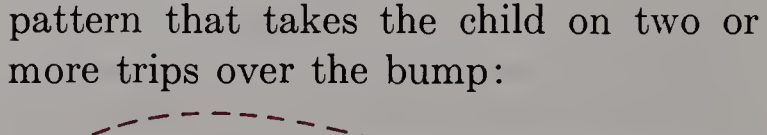
Some teachers want to hurry the idea of "place value" along. They are anticipating that rueful day when they must begin imposing on children the complicated business of carrying and borrowing.

No matter how carefully teachers explain the short cuts of carrying and borrowing, it is not nearly as rich an experience as when a pupil discovers it for himself. This discovery is adequate reward to the teacher for many hours of waiting and watching, and then leading children toward feeling a solid bump as they move out along the number line from *9* to *10*.

The Math Workshop for Children keeps the secret of "positional" or "place-value" for children to discover. However, many experiences are provided to nurture a growing sense of the pattern in the system of numeration.

$8+3=\square$ $7+10=\square$ $22-4=\square$

Moving along the number line requires only the ability to count. Taking three jumps to the right (addition) from 8 makes 11 the landing point. Taking four jumps to the left (subtraction) from 22 makes 18 the landing point. Likewise $7+10=17$. In the two addition examples, we went over a bump—from one digit to two digit numerals. And here's a little pattern that takes the child on two or more trips over the bump:




Subtract:

| | | | | | |
|---|---|---|---|---|---|
| 22 | 22 | 22 | 22 | 22 | 22 |
| 0 | 1 | 2 | 3 | 4 | 5 |
| 22 | 21 | 20 | 19 | 18 | 17 |

Add:

| | | | | | |
|---|---|---|---|---|---|
| 12 | 13 | 14 | 15 | 16 | 17 |
| 5 | 5 | 5 | 5 | 5 | 5 |
| 17 | 18 | 19 | 20 | 21 | 22 |

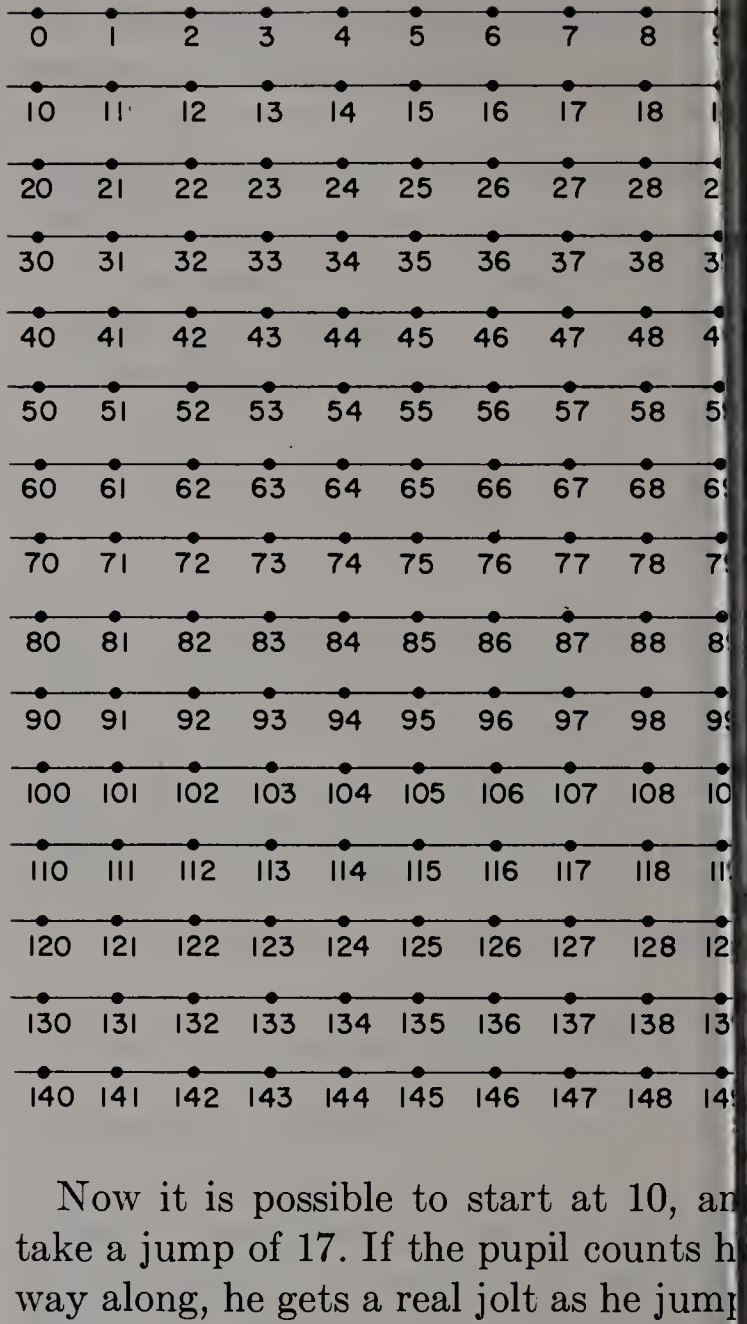
Here are three Cross Number Puzzles taken from a much longer series inserted along the way in early grade material:

| | | |
|---|---|---|
| 7 | 3 | 10 |
| 1 | 2 | 3 |
| 8 | 5 | 13 |

| | | |
|---|---|---|
| 7 | 7 | 14 |
| 3 | 5 | 8 |
| 10 | 12 | 22 |

| | | |
|---|---|---|
| 10 | 7 | 17 |
| 10 | 8 | 18 |
| 20 | 15 | 35 |

Often teachers feel hemmed in by an ordinary number line—it is too short to demonstrate that $10+17=27$. So, we have drawn a number line from 0 to 149, chopped it into convenient chunks, and arranged the chunks down the page. It looks like this:




Now it is possible to start at 10, an take a jump of 17. If the pupil counts h way along, he gets a real jolt as he jump from 19 to 20.

There's a real temptation for the chi not to count the whole way, but rath to jump from 10 to 10+10 or 20, a then count over 7 to 27. Or, getting th tedious business out of the way first, h can count from 10 to 10+7 or 17 a then jump down a row to 27.

Some child may wonder if this short cu is "fair." What does he mean? He ma not be sure he will get the "right answer. Well, there's no alternative at this poin to lots of trials. Is this jumping from ro to row a reliable time saver?

How about $17-10$? That's an easy on to solve by jumping up the page to sub tract. Is this short cut fair? What happen

one jumps up or down two rows at a me? How about 5 rows? How about rows?

In how many ways is it possible to ake the trip from 15 to 39?

a. down two rows and four jumps to the right (15+20+4)=39;

b. four jumps to the right and down two rows (15+4+20)=39;

c. two jumps to the right, down two rows, and two more to the right (15+2+20+2)=39; etc.

Now jump from 27 to 4: up two rows d three to the left, or 27−20−3=4.

As soon as a few of the ground rules e laid, games will easily get under way. 'm thinking of a number. Start at 15 d jump three rows down and four to the t. What is the number?" When some pils get really proficient, the rules can changed so that it is not fair to look at e number line chunks—each must work t the answer in his head.

Another game can be started by asking: How can you get from 15 to 15+13 in o jumps?" Once the class has proved it derstands this, variations can be introced by asking, "How can you take such trip in three jumps?" "Four jumps?" Can you make it in 10 jumps?" "Is there unlimited number of ways to jump m one number to another such as from to 15+13?"

What is the simplest way to jump from to 45−9? If necessary it can be done taking nine jumps of one each—going er the bump from 40 to 39. But it's icker to jump one row up and jump one the right: 45−9=45−10+1=36. w, get from 36 to 36−9; from 27 to −9; from 18 to 18−9; from 9 to 9−9. is pattern is, of course, repeated by a series of subtractions by nine beginning with any number; also by a series of additions by nine.

What pattern develops by a series of additions or subtractions by 11? How about repeated additions or subtractions of five?

The simplest pattern, of course, is repeated additions and subtractions by 10. And the extent to which this pattern is obvious is a measure of a growing knowledge on the part of pupils of the positional value of numbers.

By now, pupils have been equipped with three different tools for tackling "tough problems." How much is 57+15? Pupils may:

a. draw a number line and count;

b. put the numbers in cross number puzzle forms;

| | | |
|---|---|---|
| 50 | 7 | 57 |
| 10 | 5 | 15 |
| 60 | 12 | |

c. or take a trip on the page of number line chunks—down one row and five to the right, or down two rows and five to the left.

These tools enable pupils to solve "tough problems." But more important, they nurture a growing feeling of the way in which our decimal system is put together—how it works.

Working with these tools also deepens the children's understanding of the decimal system and provides opportunities for them to explore other numeration systems —other ways of naming numbers. How do

you write the name for the number of fingers on both hands if you have only five digits? Six digits? Seven digits? Can all numbers be named if we have only the two digits—0 and 1? How do the patterns of these other systems compare with the decimal system? Is any system of naming numbers "best?"

All of these experiences give pupils increased confidence when working with short cuts in computation. But the main purpose behind all of these explorations and discoveries is to cultivate a deeper understanding of the patterns and logic by which a numeration system can enable a person to write the name for every number he wants, and the next larger or lesser number.

## IV. THE BASIC FACTS AND OPERATIONS OF ARITHMETIC

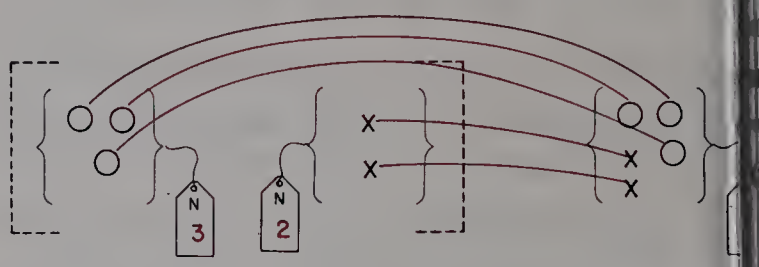
What is addition? What is multiplication? Why doesn't 1+2=12? Why does a drop of blue Easter egg dye added to a cup of water produce a cupful of blue liquid? If candy is 1¢ a piece, why do you get six pieces for 5¢? If one dumps a pail of sand on the beach and, next to it, 2 pails of sand, why is the second pile only slightly higher? If you add water to flour, why does the mixture become something quite different—paste? Why does the fastest jet plane have only one engine, while a four-engine plane is much slower? If we add water to a potted plant, why does the volume remain the same? Why is one dime worth two nickels? What do we mean when we say that rabbits multiply?

To help pupils understand the precise mathematical meaning of addition and multiplication, we must select from everyday experience those examples of addition and multiplication that are analagous mathematical addition and multiplicati

### A. Addition

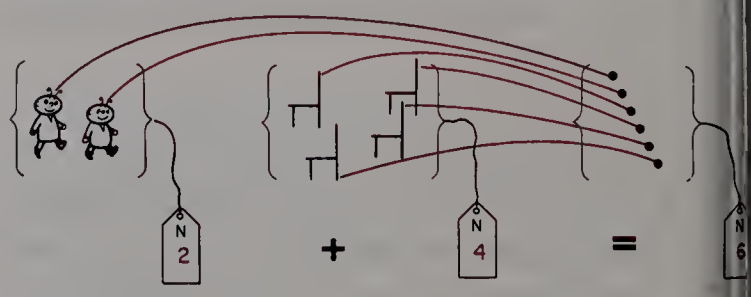
#### 1. Whole Numbers

The Math Workshop for Children e ploys a wide variety of illustrative m terial which suggests addition: counti the elements in two pictures, jumping the number line, Numo pushing sets blocks together, and matching with t numbers of counting. In the picture b low, addition is displayed in terms matching and one-to-one correspondenc

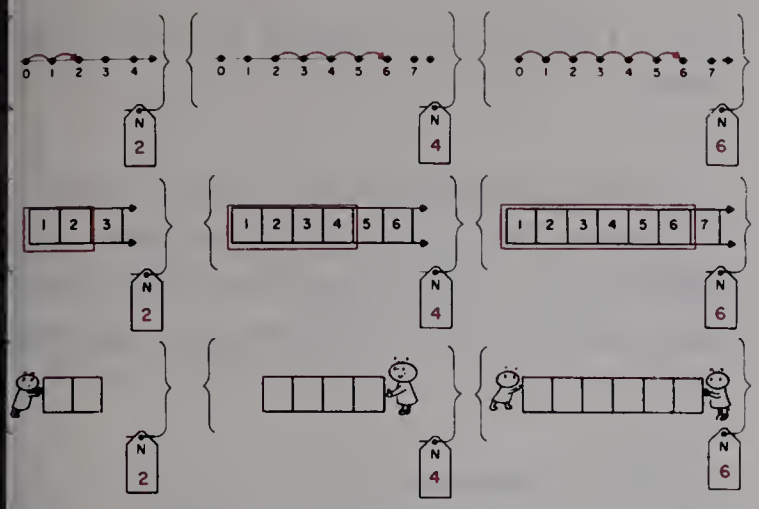



In adding, we simply consider all t objects of two separate collections as t objects of another collection—and cou the objects.

Since all that matters is the *number* the collection, we have no interest in an other property, such as shape or colo The goal is to reach an understanding i which the following illustration will n result in a feeling of frustration on th part of the children:




The mathematical "goings on" are ex expressed by: 2+4=6. And the illustra tion of the model should be the same a when number line jumps, scoreboards, o blocks are substituted:



But it is not enough that the teacher draw many pictures or diagrams to illustrate an idea. The test of success is measured by the ability of each child to conjure up a picture that suggests his own concept of addition. This picture may be drawn on paper, expressed in words, created by gestures, or shown by the manipulation of objects.

Do not hurry over this absolutely basic concept.

As addition facts are developed, it is possible to begin keeping records. First, list them in the chronological order they are discovered or noticed:

| | | |
|---|---|---|
| 1+1=2 | 2+0=2 | 1+2=3 |
| 3+1=4 | etc. | |

Next, in lists that have an obvious order:

| | | | |
|---|---|---|---|
| 0+0=0 | 1+0=1 | 2+0=2 | 3+0=3 |
| 0+1=1 | 1+1=2 | 2+1=3 | 3+1=4 |
| 0+2=2 | 1+2=3 | 2+2=4 | 3+2=5 |

Another arrangement that will suggest an important idea is the following:

| | |
|---|---|
| 1+2=3 | 2+1=3 |
| 0+3=3 | 3+0=3 |
| OR | |
| 0+5=5 | 5+0=5 |
| 1+4=5 | 4+1=5 |
| 2+3=5 | 3+2=5 |

A series of such patterns suggest the idea that in addition, order is not a factor. And here are four names for *3*, and six names for *5*. Are there others? How many such names are there for 4? Make a record. Keep score.

Other names for:

| 3 | 4 |
|---|---|
| 1+2 2+1 | 2+2 3+1 |
| 0+3 3+0 | 1+3 0+4 |
| | 4+0 |
| (total−4) | (total−5) |

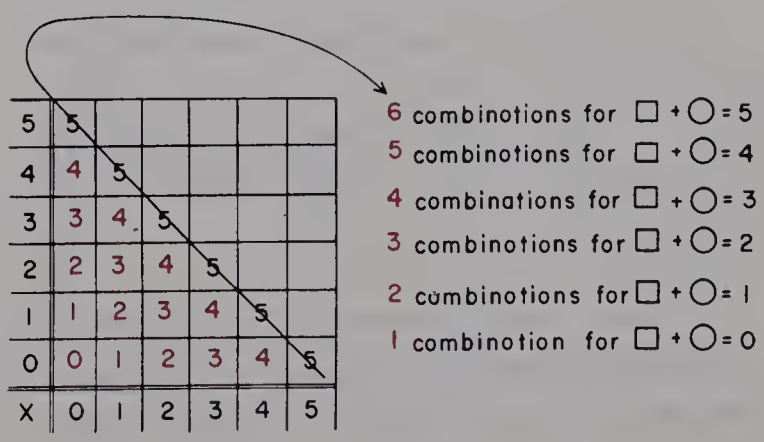
| 5 | 6 |
|---|---|
| 0+5 5+0 | |
| etc. | |

Here is still another arrangement of these other names for each number. This table gives a basis for answering the question, "are you sure you have listed them all"?

| | | |
|---|---|---|
| 0+1=1 | 0+2=2 | 0+3=3 |
| 1+0 | 1+1 | 1+2 |
| | 2+0 | 2+1 |
| | | 3+0 |

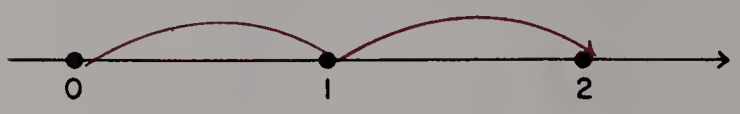
| | |
|---|---|
| 0+4=4 | 0+5=5 |
| 1+3 | 1+4 |
| 2+2 | 2+3 |
| 3+1 | 3+2 |
| 4+0 | 4+1 |
| | 5+0 |

This same collection of facts reveals its structure from still another point of view when it is recorded in the *adding machine*. Here is another basis for arguing that all possibilities have been listed.


6 combinations for □ + ○ = 5
5 combinations for □ + ○ = 4
4 combinations for □ + ○ = 3
3 combinations for □ + ○ = 2
2 combinations for □ + ○ = 1
1 combination for □ + ○ = 0
X 0 1 2 3 4 5
0 0 1 2 3 4 5
1 1 2 3 4 5
2 2 3 4 5
3 3 4 5
4 4 5
5 5

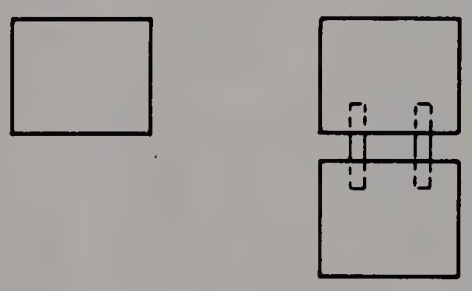
## 2. Fractions

Each of the above examples utilizes the same collection of facts. Each arrangement shows a way in which children can be encouraged to record their first discoveries in patterns which reveal interrelations. Putting the fragments back together makes the details add up to a bigger idea.




When this picture strongly suggests "two" and "two jumps of one" on the number line, the question can be raised—"Are there any numbers between *0 and 1* and between *1 and 2?*"

In first or second grade, the discussion can't go very far without some help. Three children are selected. Two are given pieces of paper—some taped together in the following manner:



Three children are involved thus far (one, of course, has not been given a piece of paper and is to suggest the idea of "zero"). These children are asked to stand in line so that they are a tableau of a number line.

If "jumping" is in order, other children can be asked: to stand at "zero;" make a jump of one; then a jump of one more; jump to one; start at two and jump back one.

Next a child is given two cookies and asked where he might stand at the number line. Another is given one, and another none. They, too, are asked to "take their places."

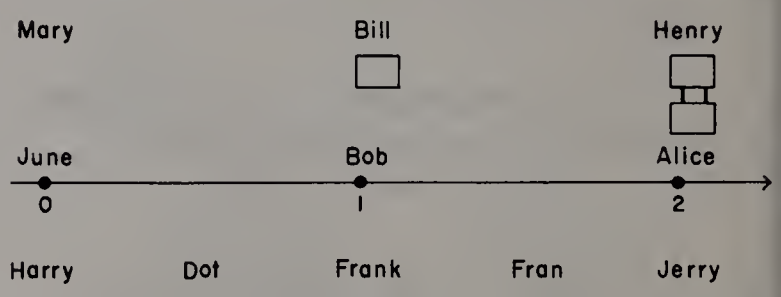
Now the stage is set to break a cookie. Give one half to a child and ask the pupils if they know where he might stand at the number line. (There is no hurry. Let everyone try out his ideas.)

Break another cookie in half. Add the half left over from the other cookie—three halves now—and give them to another child. Where should he stand?

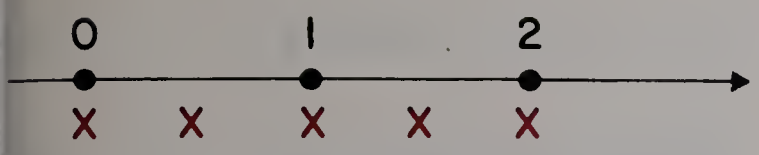
In both of these activities, the object is *to try for understanding* on a highly intuitive level. Beware of "right," "wrong," or "that's the place." Actually, if the child with one half thinks he might stand at "one" on the human number line, who is to say he is "wrong?" And if "three halves" would feel at home standing at "three," he is being utterly logical.

The usual result, however, is general agreement that "one half" would stand between zero and one; "three halves" between one and two. And when we give other children one cookie in two pieces, two cookies in four pieces, they will form a number line of half cookies—aided by another "no-half cookie" actor.

A record of the number lines can be recorded on the board as follows:



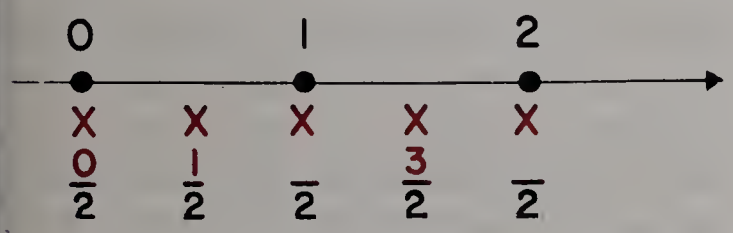

Now ask whether the number line can't be made more useful so it can take care of counting half cookies as well as whole cookies:



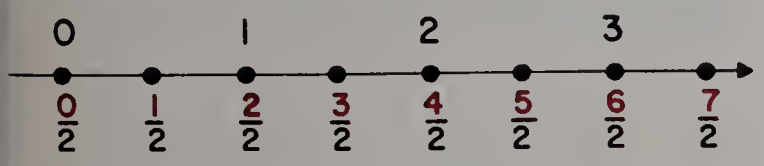

It is obvious where June and Bob and Henry would be in the number line picture. Next mark where the "half cookie" holders would stand and mark the spots with "x."

To write "zero half cookies" would get too complicated, so we ask for some kind of abbreviation. Someone usually knows "½," but if no one suggests it, it can be introduced by explaining that the numeral on the bottom shows into how many parts the cookie or apple or other object was broken or cut. The numeral on the top tells how many of those parts we are talking about.

Once this concept is understood, the question can be asked as to where the "½" should be put on the number line picture. If Fran had three half cookies, what is the best way to write this number of cookies? With a little help, "3/2" will come out, and should be written in its proper place:




Usually, children will want to write 0/2 for the place where Harry would stand, 2/2 for the place where Frank would stand, etc.:




Using cookies or other manipulative materials, the following statements can be completed:

$\frac{1}{2} + \frac{1}{2} = \frac{2}{2}$ $\frac{3}{2} + \frac{1}{2} = \frac{4}{2}$ $\frac{1}{2} + \frac{2}{2} = \frac{3}{2}$ etc.

Now the results can be recorded in a grid which the children can complete as far as it goes—and some will want to expand the grid on their own.

| | | | | | |
|---|---|---|---|---|---|
| 3/2 | 3/2 | 4/2 | 5/2 | 6/2 | |
| 2/2 | 2/2 | 3/2 | 4/2 | 5/2 | |
| 1/2 | 1/2 | 2/2 | 3/2 | 4/2 | |
| 0/2 | 0/2 | 1/2 | 2/2 | 3/2 | |
| + | 0/2 | 1/2 | 2/2 | 3/2 | |

For a time this kind of activity on number lines of halves, then number lines of thirds, and so on, can be continued. While the teacher can point out that "2/2 is another name for 1," and "4/2 is another name for 2," and "3/3 is another name for 1," it is best to establish first a real feeling of working on one kind of number line at a time.

Later on the teacher will wonder if "1/2+1/4" is worth thinking about. Then on to "1/2+1/3." By this time, pupils are used to talking about "other names" for these new fractions; they have already learned that 3/6 is another name for 1/2, and that 2/6 is another name for 1/3, And they know how to operate on a number line of sixths.

It is important to move slowly into this area of mathematics. There should be no hurry in the first and second grades. Allow time for the children to become thoroughly familiar with these new numbers—numbers that frighten many third and fourth grade children when, one day,

they "take up fractions."

Children come to school with partially formed ideas about simple fractions. The Math Workshop for Children puts these ideas to work, clarifies them, and makes them more familiar.

In many ways, we use fractions in some of the first number experiences, and keep weaving the idea in again and again as the activities expand. This is the idea of the "spiral curriculum" that is fundamental in the Math Workshop for Children.

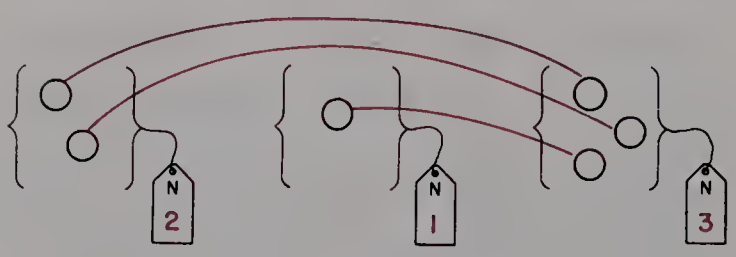
### 3. Subtraction—the Inverse of Addition

While the patterns mentioned above are limited to addition, the Math Workshop for Children encourages introduction at the outset of the idea of subtraction as the opposite of addition. The grouping below deserves some discussion at the very earliest level:

$2+1=3$ $\qquad$ $3-1=2$
$1+2=3$ $\qquad$ $3-2=1$

OR

$3-1=2$ because $2+1=3$
$3-2=1$ because $1+2=3$




This same picture can be utilized to suggest each of the eight mathematical statements above. Another picture that helps in the discussion at hand includes *two girls and one boy;*

$2+1=3$

There are two girls and one boy. There are three children all together.

$1+2=3$

One boy was joined by two girls. The three walked to school together.

$3-1=2$

There are three children. One is a boy; then two must be girls.

$3-2=1$

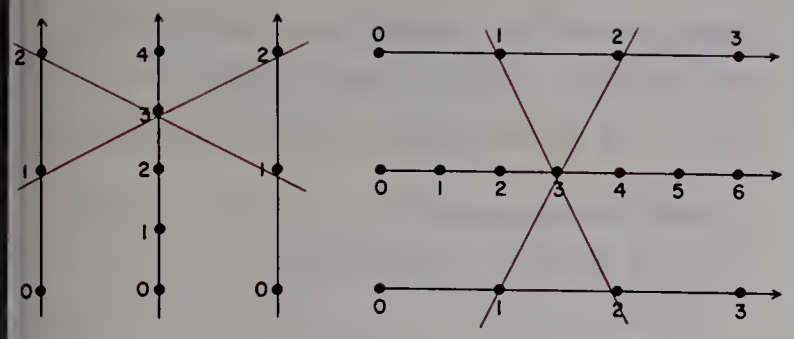
Of the three children walking to school, two are girls. There must be one boy.

By treating each addition fact of arithmetic as an example of the rule that order in addition is not a factor (commutative law) and that subtraction takes apart what addition puts together (inverse law), the teacher is illustrating two of the dozen basic principles upon which all arithmetic is based. Pointing out the interrelationship between addition and subtraction can help to cut down the number of basic facts of arithmetic to be learned.

The Math Workshop for Children employs several devices to help develop an understanding of these four versions:

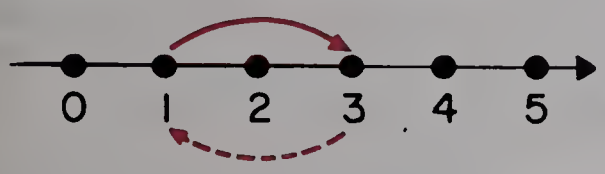
$3+2=5$ $\qquad$ $5-2=3$
$2+3=5$ $\qquad$ $5-3=2$

Adding and subtracting on a Number Ladder demonstrates the four versions graphically:



It is not necessary to specify whether the first numeral we read on the Number Ladder shall be located on the top or bottom. The same results are obtained. And, in using the diagram to talk about subtraction, we start in the middle, getting the same result if the second numeral (of 3−2) is top or bottom.

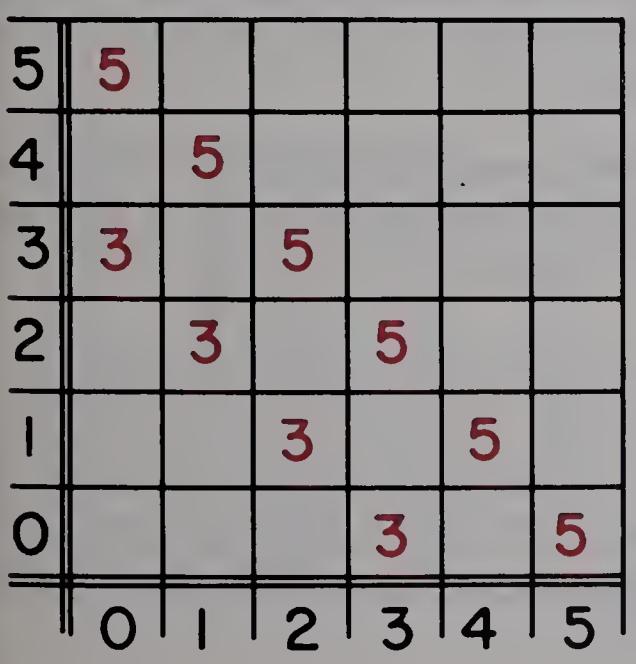
On the number line, pictures are often presented to suggest all four forms, such as:




AND

1+2=3 and 3−2=1
2+1=3 and 3−1=2

Using the Adding Machine for subtraction presents the inverse again.




In the discussion of addition it was shown that there are six names for 5 in the form of ___+___=5. And, likewise, there are six pairs of numbers to satisfy the expression 5−___=___. The generalization is quickly reached that there and N+1 names in the form of N−___=___. Again, the perfect analogy between addition and its inverse is graphically demonstrated.

In tabular form, the argument looks very much like the earlier collection of addition facts.

| | | | |
|---|---|---|---|
| 0−0=0 | 1−0=1 | 2−0=2 | 5−0=5 |
| | 1−1=0 | 2−1=1 | 5−1=4 |
| | | 2−2=0 | 5−2=3 |
| | | | 5−3=2 |
| | | | 5−4=1 |
| | | | 5−5=0 |

## B. Multiplication

### 1. Whole Numbers

Multiplication is another abstract idea that does not jump full blown into the young mind. It is a concept that is often elusive. Our concept must help us understand clearly that:

83×87=7221 3/5×175=105
2/3×5/7=10/21

The idea that multiplication is "repetitive addition" is hardly an adequate tool.

As in addition, it is helpful to have pictures which describe what multiplication is. Out of these pictorial suggestions, the child can see a common aspect that we call "multiplication."

For example: consider two groups of children—Jim, Bob, Alice, Mary, and Joan—two boys and three girls. In how many ways can a couple be chosen, one from the set of two boys and the other from the set of three girls?

Jim and Alice; Bob and Mary;
Jim and Joan; Bob and Alice;
Jim and Mary; Bob and Joan

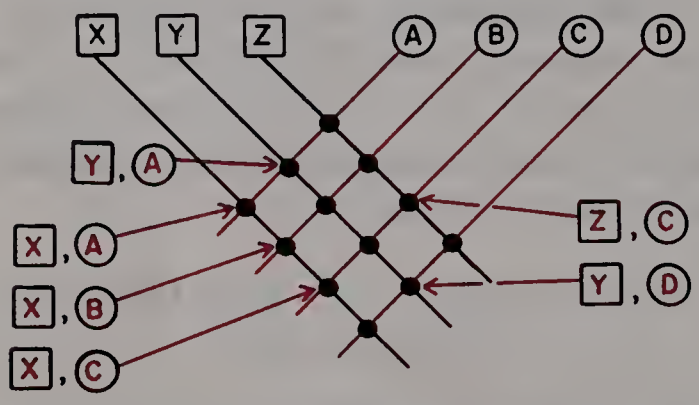
There are *six* ways; and that example can be one very clear picture of what we meant by:

$$2\times3=6$$

Now, the interesting thing about this first look at "multiplication" is that it didn't require any visual picture to define the idea. The example given—a mental picture of boys and girls—could be done over the telephone, with no gestures or diagrams involved.

The only trouble may be that the pupils don't recognize it as "multiplication." Does it always work?

Draw a picture on the board this time. Use 3 squares and 4 circles, and pair each square with each circle. (The shapes are lettered so they can be talked about.)




If we label all the points each will represent a pairing between a square and a circle. Further investigation quickly reveals that no pair is repeated and none is left out. The pupils can count the number of pairings. They will find there are 12 because:

$$3\times4=12 \quad \text{and} \quad 4\times3=12$$

This approach is particularly important because it permits children to "discover multiplication." First graders can act out the choosing of couples from a group of boys and girls, varying the size of the groups and recording the results. This is not merely counting nor repetitive addition. It's a new idea—multiplication.

*Jim* and *no girl*—
no couples $(1\times0=0)$
*Jim* and *Alice*—
one couple $(1\times1=1)$
*Jim* and *Alice and Mary*—
two couples $(1\times2=2)$
*Jim and Bob* and *Alice and Mary*—
four couples $(2\times2=4)$

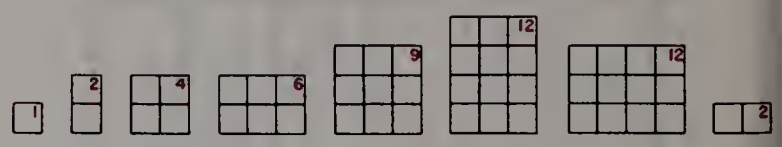
If there is a blue blouse and a red blouse for a paper doll, and a red skirt and a blue skirt, how many different outfits can be arranged?

What happens if a black skirt is added to the list? Is there any situation in which there would be only three possible outfits? (Yes, one blouse and three skirts or three blouses and one skirt.)

But presenting a single picture of multiplication to children is apt to lead to a confused identification of the abstraction with the picture. Other kinds of examples must be used.

From a piece of graph paper (one inch squares, perhaps), cut rectangles of different sizes and shapes. After a bit of cutting, pick out the biggest and ask if you have cut all the possible rectangles that are smaller than the one you are holding.

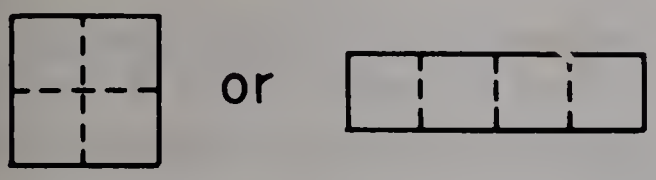
Now count the squares in each rectangle and record the results in the upper right corner:




By turning the blocks face down, the stage is set for a guessing game. How many squares? Turn the block up to check results.

Then the game shifts to, "I'm thinking of one of the paper blocks. It has nine squares on it. Can you point to it"?

This leads to problems of similarity with two rectangles both of which contain four squares. Which one?



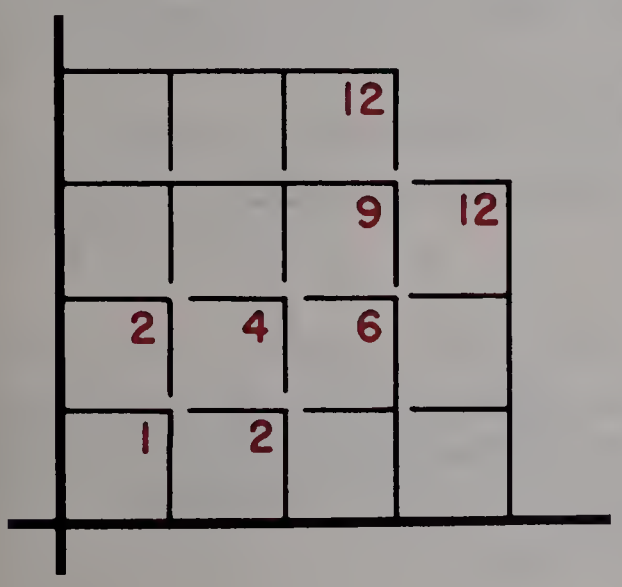

Attention is then focused on the length of the sides. Each edge of the block is made up of edges from a number of blocks. And now the game shifts to guessing "which one" by talking about the edges.

A record can now be made to summarize the results.

| No. of small box edges | | No. of small boxes |
|---|---|---|
| Along the top | Along the side | |
| 1 | 1 | 1 |
| 1 | 2 | 2 |
| 2 | 2 | 4 |

This record, of course, is then compared with the record of couples—a boy and a girl—from groups of various sizes.

Don't throw away the rectangular paper blocks or squares. Paste them in a corner with the smaller blocks pasted on top of the larger blocks as shown below.




Now it is possible to answer the question about whether or not all the possible shapes smaller than the largest are present. Where there is a small box without a numeral in it, a rectangular shape is missing—and ought to be added, because as the structure grows it becomes a multiplying machine. Patterns begin to jump out—the "multiplication tables"* that some have tried to banish from children's view.

Keep this table, because we will return to it later.

Now for an approach that is the repetitive addition idea in disguise. Professor W. W. Sawyer has aptly called it the "wallpaper arithmetic" approach.

Below is a section of a number grid on which the pupil is asked to color in every third square. (Surprisingly enough, there is almost no difficulty in getting from 9 to 12—"around the corner," as one child put it.)

A very simple pattern in color develops. Different patterns emerge when the pupil is asked to color every second, fourth, or fifth square. Each time, a record, such as the following, is kept of the results:

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |

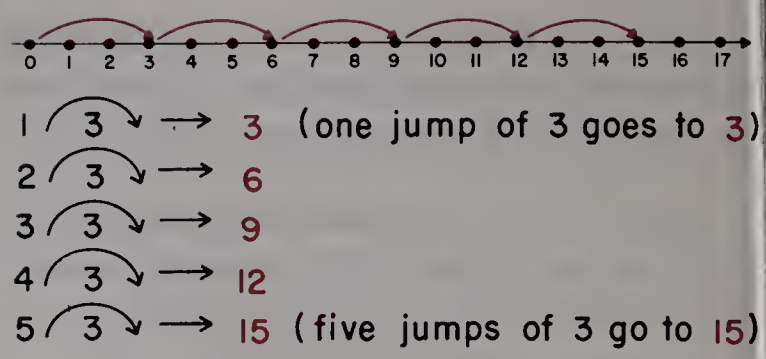
| | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th | 8th | 9th | 10th |
|---|---|---|---|---|---|---|---|---|---|---|
| NUMBERS COLORED | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 |

| | | | | |
|---|---|---|---|---|
| 4 | 4 | 8 | 12 | 16 |
| 3 | 3 | 6 | 9 | 12 |
| 2 | 2 | 4 | 6 | 8 |
| 1 | 1 | 2 | 3 | 4 |
| X | 1 | 2 | 3 | 4 |

* See page 37 for the 100 facts of multiplication arranged in a ten-by-ten grid.

Now get out the table that was constructed of rectangles. Each "wallpaper arithmetic" pattern is exactly the same as one of the columns and one of the rows in the table.

Now on to a fourth picture of multiplication, jumping on the number line. A picture of "jumps of 3" would look like this: (A descriptive notation is used—one initiated by this program.)



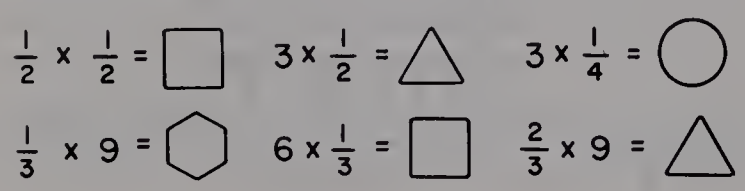

Here again, the resulting patterns can be viewed in such a way that they are identical with those in each previous activity.

| Row | Column | Where Row and Column Cross |
|---|---|---|
| Number of girls | Number of boys | Number of couples |
| Number of blocks along sides of rectangle | Number of blocks along top of rectangle | Number of small boxes |
| Number of steps from one block to the next | Number of blocks colored | Last number colored |
| Size of jumps on the number line | Number of jumps | Landing point on number line |

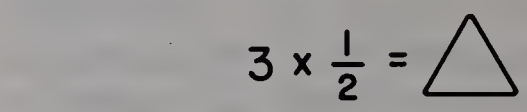
What is similar about all these different approaches? It is the abstract idea we call "multiplication." And it is a big idea that children can begin to grasp in all its great generality only as they rediscover it in many different experiences.

## 2. Fractions

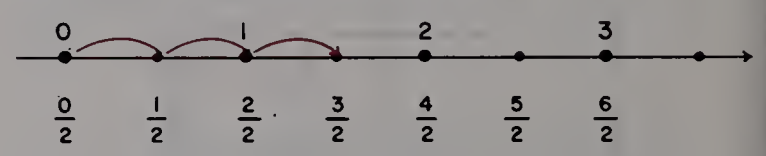
Early along the way, it is hoped some child will wonder, "Can we multiply fractions?" If this hope is not fulfilled, the teacher may be forced to initiate the discussion by asking the question: "What would happen if we tried to multiply fractions? Whatever could that mean? What would an example look like?"

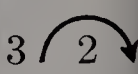
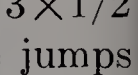



What could these strange incomplete sentences mean? Do any of them have a familiar ring of any kind? Take them one at a time. Perhaps:




Several pictures have been used to suggest multiplication, so draw those and show that they apply in this case. Start with the number line.




Since 3×2 might be seen as 3  (three jumps of 2 to the right); 3×1/2 might be seen as 3  (three jumps of 1/2 to the right). Therefore, you would land at 3/2.

Perhaps the idea will become clearer by setting up a table. The pupils will agree that:

$$3 \times 2 = 2 + 2 + 2 = 6$$

$$3 \times 1 = 1 + 1 + 1 = 3$$

$$3 \times \frac{1}{2} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$$

seems to make sense. Translated into a situation involving money, this would mean:

3 piles of coins with $2 in each would be worth $6.

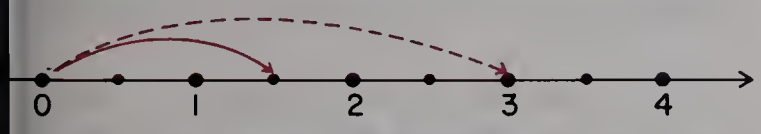
3 one dollar bills would be worth $3.

3 half-dollar pieces would be 3/2 dollars or $1.50.

Further, the pupils would like to be sure that:

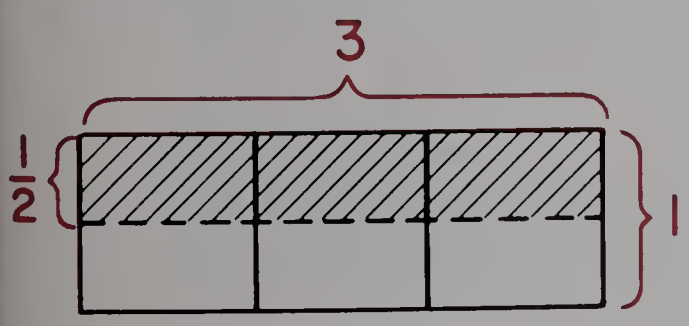
$$3 \times 1/2 = 1/2 \times 3,$$

because it certainly would help if these new numbers would behave like the old ones. In this way, the teacher does not have to start all over with new rules about the order of multiplying numbers. How might this look on a number line?




If a pupil is asked to take one half of a jump to 3, he would land halfway between 1 and 2. This is 1 plus half the distance between 1 and 2. On a number line labeled in halves, that would be "3/2."

Draw a picture of a rectangle, to suggest $1 \times 3$. Then shade an area that might suggest $1/2 \times 3$:




Again, a picture emerges which suggests that $1/2 \times 3 = 3/2$.

Follow the procedure of drawing pictures to explain each of the examples. As they analyze each picture, the children will be using tools they previously learned to use in other situations. This is an environment that permits and promotes discovery! It helps the children to gain confidence that the tools they already have at their command can be brought to bear effectively on a new and strange situation. Now the following seem completely logical:

$3 \times 1/2 = 3/2 = 1 + 1/2$
$3 \times 1/4 = 3/4$
$1/3 \times 9 = 9/3 = 3$
$6 \times 1/3 = 6/3 = 2$

But, what can be meant by:

$1/2 \times 1/2 =$ $1/2 \times 1/4 =$
$1/2 \times 2/3 =$

Again resort to tools the children already know how to use. Here are some patterns—extending from the known out toward the unknown.

$4 \times 1/2 = 4/2 = 2$
4 half circles look like

$2 \times 1/2 = 2/2 = 1$
2 half circles look like

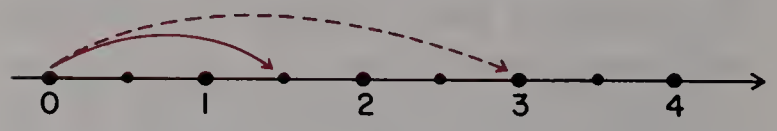
$1 \times 1/2 = 1/2 = 1/2$
1 half circle looks like

$1/2 \times 1/2 =$
1/2 of a half circle looks like _____

How much can the pupils guess about $1/2 \times 1/2$? As the first or second factor becomes smaller, does the product grow larger or smaller? Does it seem reasonable that $1/2 \times 1/2$ is less than 1? . . . less than 1/2? Each first or second factor is one-half the one above it in the table. Is each

product one-half the one above it? Ask the children to study the picture. What could one-half of one-half mean?

How about jumps on the number line?

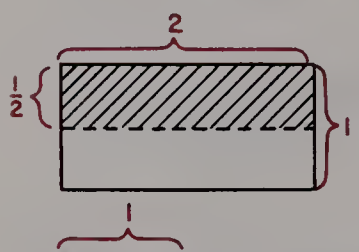



The longer line suggests 1/2 of a jump to 2.

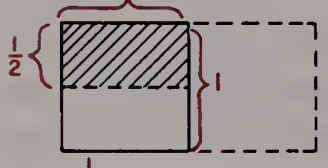
The shorter line suggests 1/2 of a jump to 1.

Draw a line suggesting 1/2 of a jump to 1/2.

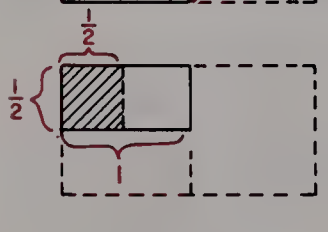
Now, draw a series of rectangular pictures.




Suggests that $\frac{1}{2} \times 2 = \frac{2}{2} = 1$




Suggests that $\frac{1}{2} \times 1 = \frac{1}{2}$




Suggests that $\frac{1}{2} \times \frac{1}{2} = \boxed{\frac{1}{4}}$

Again, every time familiar pictures of the idea of multiplication are applied, they lead to the conclusion that $1/2 \times 1/2 = 1/4$.

Patiently encourage the children to study each of the following examples of the multiplication of fractions. They will discover that:

$$1/2 \times 1/4 = 1/8$$
$$1/2 \times 1/2 = 1/4$$
$$1/2 \times 3/4 = 3/8$$

Are there any interesting conclusions suggested in the array?

Obviously, in each case they would g the correct answer if they "multiply t numerators and multiply the denomin tors," but let the children discover th as they work through many examples.

At the end of each exercise, encoura the children to test any result or patte they come upon. Is the result reasonabl Does it resemble patterns with whic they are familiar? Gradually, they w develop a feeling that what they are doi makes real sense. They will achieve growing facility in carrying out this bus ness of "multiplying numerators and d nominators."

Most important, they will understar clearly what they are doing. They w develop a technique which they can appl when they come to $3\text{-}1/2 \times 1\text{-}7/8 = \square$ They will have the mathematical knov how to tackle any situation and to fin their own way to a reasonable conclusio

### 3. Division–The Inverse of Multiplicatio

Division is the inverse of multiplicatio and can be interpreted in terms of eac example used to illustrate multiplicatio

For example, if the number of coupl that can be chosen from a group of gir and a group of boys is known, it is possib to find out how many girls are in th group by knowing the number of boys But it's more fun and a more meaningfu experience to decide how many differer groupings of boys and girls would perm the choice of 12 couples, for example.

| Boys | Girls |
|---|---|
| 1 | 12 |
| 2 | 6 |
| 3 | 4 |
| 4 | 3 |
| 6 | 2 |
| 12 | 1 |

In another example, if a rectangle of blocks, pasted up to form a grid, has "12" written in the corner, what is its shape? How many such blocks can there be in the grid?

Further, how many "wallpaper arithmetic" patterns include 12 as one of the blocks filled in? Which ones?

Finally, in how many ways can the pupils jump from 12 to 0 on the number line if, on each trip, they must continue taking jumps the same size as the first—and land squarely on 0?

Division is not a big idea at all. All its glory is reflected from, and its importance ends with, the fact that it is the inverse of multiplication. As teachers handle it in this way, many of the mysteries and much of the discomfort children commonly feel about "division" will never develop.

As one child slyly put it: "Division problems are just sneaky ways of finding out how much we know about multiplication."

### a. Division Algorisms

It took the finest mathematicians centuries to devise a simple short cut to performing such divisions as $37250 \div 215 =$ _____. Slowly, painfully, step by step, a means was evolved that, today, is part of the mathematical heritage we pass on to children in the elementary schools.

Teachers are, sometimes, so anxious to reveal this important technique with its beautiful simplicity that they hurry into presenting the division algorism to children as a series of steps to be memorized. The price exacted is a lack of understanding, that children may never overcome.

Take the time to play a series of division games which focus the spotlight on meaning. The following games have a touch of the puzzle, and they provide useful drill. Moreover, they gradually lead toward both the "subtractive" and "traditional" division algorisms. These games assume that all children would understand that each of the following statements is true:

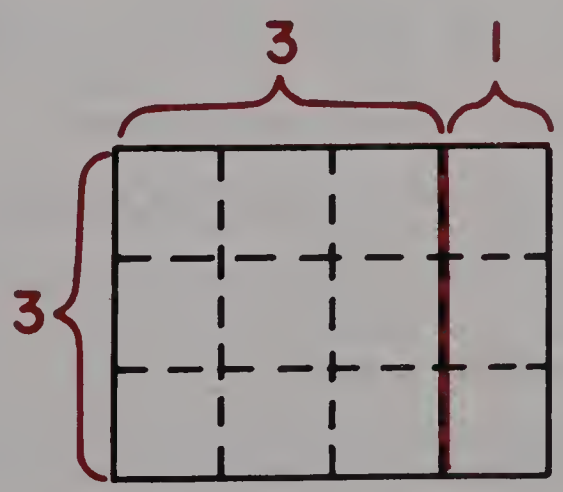
$$3\overline{)12} = 3\overline{)9+3}^{\,3+1} = 4$$

$$\frac{12}{3} = \frac{9+3}{3} = \frac{9}{3} + \frac{3}{3} = 3 + 1 = 4$$

$$12 \div 3 = (9 \div 3) + (3 \div 3) = 3 + 1 = 4$$

Is that a big assumption? If so, better stop right there. If the pupils do not understand every one of those steps, they have no basis upon which to build an understanding of the division algorism.

Discuss problems like the following: "I have to divide 12 pieces of fruit among 3 children. If I have 9 apples and 3 pears, what would be a good way to do it?" Each child, of course, gets 3 apples and 1 pear; 4 pieces in all. Now the principle must be brought out. How is it possible to break a group of 12 pieces of fruit into two groups? How does the picture below suggest what was done?



If there is any uneasiness about the example above, don't hurry! Try dozens like it. Make up stories. Draw pictures. Go back to manipulative material!

The children must have a deep understanding of the fundamental relation between addition and multiplication along with their inverses, subtraction and division, because in the algorism of division they will have to use all four operations. They will multiply and subtract, sometimes over and over again; and finally do a bit of addition—to get a quotient!

Each of the division games that follows contains but a few examples of problems that should be tackled. This series is based on steps that most children can understand completely, and is put together for the purpose of illustrating direction and movement toward the algorisms of division.

A game is invented or selected as soon as there is agreement on a rule or set of rules. In the games to follow, the special rule for each game is stated.

Game I. The rule is that the player may use no term in the dividend that has more than a single digit.

An example which violates the rule is:

$$2\overline{)12} = 2\overline{)10+2}$$

because "10" is a term in the dividend and it has 2 digits. The rule limits us to single-digit terms in the dividend.

$2\overline{)12} = 2\overline{)8+4} = 2\overline{)4+2+6} = 2\overline{)6+6+6-6}$

$3\overline{)12} = 3\overline{)\_+\_} = 3\overline{)\_+\_+\_} = 3\overline{)\_+\_-\_+\_}$

$4\overline{)12} = 4\overline{)\_+\_} = 4\overline{)\_-\_+\_} = 4\overline{)\_-\_+\_-\_}$

$6\overline{)12} = 6\overline{)\_+\_} = 6\overline{)\_+\_+\_} = 6\overline{)\_+\_-\_-\_}$

Game II The rule is that no term in the quotient may have more than a single digit:

$2\overline{)24} = 2\overline{)12+12} = 2\overline{)8+8+8} = 2\overline{)6+6+6+6}$

$3\overline{)24} = 3\overline{)21+3} = 3\overline{)\_+\_+\_} = 3\overline{)\_+\_+\_-\_}$

$4\overline{)24} = 4\overline{)\_+\_} = 4\overline{)\_+\_-\_} = 4\overline{)12+9-6+9}$

$6\overline{)24} = 6\overline{)\_+\_} = 6\overline{)\_-\_+\_} = 6\overline{)24+24+24-48}$

Game III. Same rule as II, except that the quotient is to have as many odd terms as possible

$2\overline{)36} = 2\overline{)18+18} = 2\overline{)14+10+12} = 2\overline{)10+10+10+6}$

$3\overline{)36} = 3\overline{)\_+\_} = 3\overline{)\_+\_+\_} = 3\overline{)\_+\_-\_+\_}$

$4\overline{)36} = 4\overline{)\_+\_} = 4\overline{)\_+\_-\_} = 4\overline{)\_+\_+\_-\_}$

$6\overline{)36} = 6\overline{)\_+\_} = 6\overline{)\_-\_-\_} = 6\overline{)\_-\_-\_-\_}$

Game IV. No restrictions:

$2\overline{)48} = 2\overline{)24+24} = 2\overline{)40+10-2} = 2\overline{)20+20-10+18}$

$3\overline{)48} = 3\overline{)\_+\_} = 3\overline{)\_-\_+\_} = 3\overline{)\_-\_+\_-\_}$

$4\overline{)48} = 4\overline{)\_+\_} = 4\overline{)\_-\_-\_} = 4\overline{)\_-\_-\_-\_}$

$6\overline{)48} = 6\overline{)\_+\_} = 6\overline{)\_+\_\ \_} = 6\overline{)\_+\_+\_+\_}$

Game V. All but one of the terms in the quotient must be multiples of 10: (The teacher has a decision to make: to develop either the "subtractive" or the "traditional" algorism—or both. If the "subtractive" is selected, Game V is omitted. However, both algorisms are so basically alike that most children will be able to handle both as they develop the general technique.)

$$6\overline{)72} = 6\overline{)\underset{10}{60}+\underset{2}{12}} = 6\overline{)\underset{20}{120}+\underset{0}{0}-\underset{8}{48}} = 6\overline{)\underset{10}{60}+\underset{0}{0}+\underset{10}{60}-\underset{8}{48}}$$

$$\overline{)72} = 4\overline{)\quad+\quad} = 4\overline{)\quad-\quad+\quad} = 4\overline{)\quad-\quad-\quad+\quad}$$

$$\overline{)72} = 3\overline{)\quad+\quad} = 3\overline{)\quad+\quad+\quad} = 3\overline{)\quad-\quad+\quad+\quad}$$

$$\overline{)72} = 2\overline{)\quad+\quad} = 2\overline{)\quad-\quad-\quad} = 2\overline{)\quad-\quad-\quad-\quad}$$

Game VI. No restrictions:

$\overline{)352}$ $12\overline{)294}$ $22\overline{)242}$ $50\overline{)550}$

What methods were employed once most of the restrictions were lifted? (The printed form itself is a restriction requiring decomposition of the dividend into two terms.)

Each method used must be carefully explained by the player and thoroughly discussed by the class. Every technique deserves its day in class. Any method that always leads to a correct result is a correct method.

The basis has been laid for some to go on alone and to make important discoveries as they try their techniques on such tough examples as those to follow.

Meanwhile time is needed to lead others over the hurdles they couldn't clear in Game IV or earlier.

Game VII. No restrictions at all! (Except that each player should be sure he understood Game IV and VI clearly.)

$25\overline{)1225}$ $81\overline{)5427}$

$37\overline{)296}$ $13\overline{)100}$

$25\overline{)2575}$ $7\overline{)1001}$

$12\overline{)1164}$ $317\overline{)6357}$

By this time, individual differences will have the class scattered at rough spots along the road. Some will never get to Game VII without the help of the algorism. However, the distance each has come along the way and the further advance he will make with help will contribute understanding and lend legitimacy when the algorism the teacher prefers is finally presented.

Each game should, of course, be played with many, many examples. When children are encouraged to make their own examples, they usually vary the size of the numbers according to their interests and abilities. Some catapult themselves right into a complete mastery of a satisfactory algorism.

The games cannot fail to develop and strengthen those underlying connections between the basic operations that make any division algorism possible.

### b. Division of Fractions

Allan Funt created hilarity on "Candid Camera" by stopping people on the street and confronting them with the following problem:

$$\frac{7}{8} \div \frac{2}{3} = ?$$

The responses given revealed completely weird modes of thought and ridiculous conclusions. Here, certainly, is an example from the "grey area of arithmetic."

Much of the difficulty involved in the division of fractions comes from the fact that real-life illustrations become increasingly difficult to find. How shall a teacher help pupils to discover for themselves the secret of this abstraction? One very fruitful approach is to try to discover whether someone else's discovery is sound.

The teacher might say—

"I heard Mary Burns argue once that in division, you can invert the divisor and multiply. What do you think she meant? Is her statement a true one? Will it always lead to a correct result?"

Now the work begins. Start with some examples of division, and then begin to test Mary's rule by applying it:

$8 \div 2 = \square$ $\qquad$ $\frac{7}{8} \div 1 = \square$

$\frac{1}{2} \div 2 = \square$ $\qquad$ $2 \div \frac{1}{2} = \square$

$\frac{3}{4} \div \frac{3}{4} = \square$ $\qquad$ $\frac{1}{2} \div \frac{1}{4} = \square$

Before applying the rule to these examples, we must understand exactly what Mary meant. Part of her rule is "invert the divisor . . ." To test the rule, we must know what we mean by "divisor." In the examples of division above, the divisor is the second term. It is the number by which we are dividing. If examples are written in the following form:

$2\overline{)12}$ $\qquad$ $7\overline{)49}$ $\qquad$ $11\overline{)121}$

then "2," "7," and "11" are the divisors.

If, in the statement that "$8 \div 2 = \square$," "2" is the divisor, what does "invert the divisor" mean? Mary says that the first step is to express the divisor as a fraction if it is not already in that form, for example:

| | |
|---|---|
| 2 = 2/1 | 3-1/3 = 10/3 |
| 1 = 1/1 | 3/8 = 3/8 |

Now, the second step is the "inverting" process in which the numerator and denominator change positions:

2/1 inverted is changed to 1/2
10/3 inverted is changed to 3/10
1/1 inverted is unchanged—1/1
3/8 inverted is changed to 8/3

These are real changes since 2 certainly larger than 1/2; 3/8 is smal than the inverted form of 8/3. Only 1 remains unchanged in the process.

The rest of "Mary's Rule" tells us multiply. Somehow, multiplying with t inverted divisor is the same as dividing the original divisor, according to Ma

To put "Mary's Rule" to the test, t it in as many different cases as possib Does it always yield a reasonable answe Is there even one example in which leads to trouble? Examine a series statements that have a special patte (a whole number divided by a wh number). Apply "Mary's Rule" in ea case.

$8 \div 2 = 8 \div \frac{2}{1} = 8 \boxed{\times \frac{1}{2}} = \frac{8}{2} = 4$

$100 \div 4 = 100 \div \frac{4}{1} = 100 \boxed{\times \frac{1}{4}} = \frac{100}{4} = 2$

$144 \div 12 = 144 \div \frac{12}{1} = 144 \boxed{\times \frac{1}{12}} = \frac{144}{12} = 1$

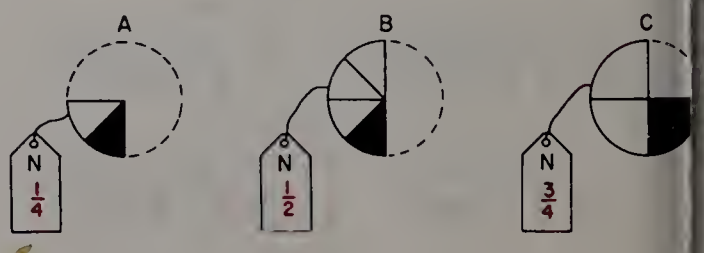
So far the rule works. Perhaps anoth pattern (a fraction divided by a wh number) will turn up questions. Are the results reasonable?

$\frac{1}{4} \div 2 = \frac{1}{4} \div \frac{2}{1} = \frac{1}{4} \boxed{\times \frac{1}{2}} = \frac{1}{8}$

$\frac{1}{2} \div 4 = \frac{1}{2} \div \frac{4}{1} = \frac{1}{2} \boxed{\times \frac{1}{4}} = \frac{1}{8}$

$\frac{3}{4} \div 3 = \frac{3}{4} \div \frac{3}{1} = \frac{3}{4} \boxed{\times \frac{1}{3}} = \frac{1}{4}$

Pictures may make it clear wheth these results are correct.



Picture "A" represents 1/4. Cut it into pieces, with one piece shaded. The haded piece certainly represents 1/8, if he tagged part is a picture of 1/4:

$1/4 \div 2 = 1/8$

Picture "B" represents 1/2 cut into 4 arts with one shaded. If that is true, hen the shaded area represents 1/8:

$1/2 \div 4 = 1/8$

Picture "C" represents 3/4 cut into 3 ieces; the shaded piece represents 1/4:

$3/4 \div 3 = 1/4$

So "Mary's Rule," applied to these xpressions and to the pictures which lustrate them, leads to results that seem easonable.

Another familiar group of expressions as a pattern in which each statement is n example of a number divided by itself:

$$\frac{3}{1} \div \frac{3}{1} = \frac{3}{1} \times \frac{1}{3} = \frac{3}{3} = 1$$

$$\frac{1}{2} \div \frac{1}{2} = \frac{1}{2} \times \frac{2}{1} = \frac{2}{2} = 1$$

$$\frac{3}{4} \div \frac{3}{4} = \frac{3}{4} \times \frac{4}{3} = \frac{12}{12} = 1$$

In this group it is obvious that the rule orks for all the examples. Pupils will member any number "divided by itself" another name for "1." "Mary's Rule" supported by this familiar conclusion.

The same is true of another group. In is pattern are examples of numbers vided by one. If "Mary's Rule" operates roperly on such examples, the dividend ill be left unchanged.

$5 \div 1 = 5 \div 1/1 = 5/1 = 5$

$1/3 \div 1 = 1/3 \div 1/1 = 1/3$

$7/8 \div 1 = 7/8 \div 1/1 = 7/8$

The rule has passed another hurdle.

Try another pattern consisting of examples in which the divisor is a fraction, one that is not a whole number in fraction form.

$$2 \div \frac{1}{2} = 2 \times \frac{2}{1} = 4$$

$$1 \div \frac{1}{3} = 1 \times \frac{3}{1} = 3$$

$$5 \div \frac{3}{4} = 5 \times \frac{4}{3} = \frac{20}{3} = 6\frac{2}{3}$$

The first example might be explained by asking, "How many halves are there in "2?" Or, how many half dollars are there in $2? Or, how many half-hour class periods are there between 9 o'clock and 11 o'clock? In every case, the conclusion agrees with the results of using "Mary's Rule."

The next example suggests a situation such as dividing a foot ruler into thirds and finding out how many pieces there are. One foot is 12 inches, and 1/3 of a foot is four inches. Stated this way, the example does not need to involve fractional forms.

$12 \div 4 = 3$

The third example is more of a challenge. Applying the rule results in the following:

$5 \div 3/4 = 5 \times 4/3 = 20/3 = 6\text{-}2/3$

Trying to express the result in terms of money, the teacher could ask, "If I had $5 and wanted to find out how many stacks I could make worth 75¢ each, what would I do? A clever way to start out would be to change the $5 into quarters. I would have 20 quarters. Then, I would make stacks of 3 quarters each—so each stack would be worth 75¢. I would find that I could make 6 piles and have 2 out of the 3 I needed for a seventh pile."

Not only is this the same result, but strangely enough "20" and "3" turned up

in the teacher's explanation as well as in the original example.

So, we go on to another pattern—one fraction divided by another—and we have used the rule again, testing it.

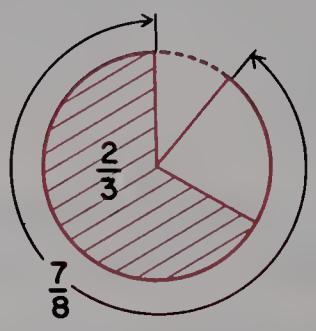
$1/2 \div 1/4 = 1/2 \times 4/1 = 4/2 = 2$

$3/4 \div 1/4 = 3/4 \times 4/1 = 12/4 = 3/4 = 3$

$7/8 \div 2/3 = 7/8 \times 3/2 = 21/16 = 1\text{-}5/16$

How many 1/4's in 1/2? The result of "2" seems reasonable. There are four 1/4's in one, so there ought to be two 1/4's in 1/2. And, it seems reasonable to say that there are three 1/4's in three-fourths.

But $7/8 \div 2/3 = 21/16$ or 1-5/16! How many 2/3's are there in 7/8? Draw a picture of 7/8, then shade in a picture of 2/3. The pupils can see that there is something more than one of them—and there might be 1-5/16 of them. But this is not a very clear picture.




However, the pupils will be willing to accept this example with the feeling that the results are at least close.

For a moment, forget "Mary's Rule" and look at "division" from a different angle. This may in turn shed light on "Mary's Rule."

The pupils will agree that the following statements are all true.

$8 \div 8 = 1 \quad 8 \div 4 = 2 \quad 8 \div 2 = 4 \quad 8 \div 1 = 8$

$4 \div 8 = 1/2 \quad 4 \div 4 = 1 \quad 4 \div 2 = 2 \quad 4 \div 1 = 4$

$2 \div 8 = 1/4 \quad 2 \div 4 = 1/2 \quad 2 \div 2 = 1 \quad 2 \div 1 = 2$

$1 \div 8 = 1/8 \quad 1 \div 4 = 1/4 \quad 1 \div 2 = 1/2 \quad 1 \div 1 = 1$

In a grid display, these known facts would look like this:

| Dividends \ Divisors | 8 | 4 | 2 | 1 |
|---|---|---|---|---|
| 8 | 1 | 2 | 4 | 8 |
| 4 | 1/2 | 1 | 2 | 4 |
| 2 | 1/4 | 1/2 | 1 | 2 |
| 1 | 1/8 | 1/4 | 1/2 | 1 |

The divisors, 8, 4, 2 and 1, are written across the top of the grid. They form a series in which each term is 1/2 of its predecessor. This series could be extended forever—8, 4, 2, 1, 1/2, 1/4, 1/8, 1/16 etc.

Dividends have been chosen from the same series. The rows and columns of results seem like sections out of the same series.

Suppose the divisor and dividend series are extended to make the grid larger:

| Dividends \ Divisors | 8 | 4 | 2 | 1 | 1/2 | 1/4 | 1/8 | 1/16 |
|---|---|---|---|---|---|---|---|---|
| 8 | 1 | 2 | 4 | 8 | | | | |
| 4 | 1/2 | 1 | 2 | 4 | | | | |
| 2 | 1/4 | 1/2 | 1 | 2 | 4 | | | |
| 1 | 1/8 | 1/4 | 1/2 | 1 | | | | |
| 1/2 | | | | 1/2 | 1 | | | |
| 1/4 | | | | 1/4 | | 1 | | |
| 1/8 | | | | 1/8 | | | 1 | |
| 1/16 | | | | 1/16 | | | | 1 |

The Column under divisor "1" can be filled in readily because any number remains unchanged when divided by "1." The diagonal downward to the right is likewise easy to fill in because any number divided by itself is "1."

For experimental purposes, select a square at random and make it a part of both patterns. For example, under "1/2" and in row "2," the patterns indicate there probably ought to be a "4" because (a) that square is on a diagonal that has only "4's" thus far, and (b) it is an extension of the series 1/4, 1/2, 1, 2 . . .

Putting a "4" in the grid seems to mean that:

$$2 \div 1/2 = 4$$

Is this a true statement?

1. The extension of both patterns would argue that "4" ought to be in that square.
2. Mary Burns would apply her rule $2 \div 1/2 = 2 \times 2/1 = 4$ and she would agree.
3. The result is the same as the answer to the question, "How many half-dollars are there in $2?"

In fact, as one square after another is filled in, these same three arguments will support each answer that fits the pattern of the grid. (If some pupils need further enlightenment, still another example can be used. Because of the particular numbers employed in this grid, the teacher can revert to the ruler with its division into 1/2's, 1/4's, 1/8's, 1/16's, and 1/32's. Eight inches divided into 1/16's produces $8 \times 16$ units, or 128 units, the same as $8 \div 1/16 = 128$.)

Has it been proved that, "In division, you can invert and multiply?" Surely not. What has been demonstrated so far, however, is that "Mary's Rule" leads to no conclusions that are contrary to our knowledge of arithmetic, none that are meaningless.

The pupils should, of course, be led to try several other approaches before they are satisfied that Mary has hit upon a reliable rule.

The teacher can argue that division is the inverse of multiplication. This means that:

$6 \div 2 = 3$ because $3 \times 2 = 6$
$8 \div 1/2 = ?$ because $\_\_ \times 1/2 = 8$

The last part of the second statement requires that a "16" be put in to read:

$16 \times 1/2 = 8$

So, it must be true that "$8 \div 1/2$" equals "16." The same result is obtained when "Mary's Rule" is used.

The same principle can be demonstrated in a different form.

$8\overline{)8}^{1}$ $4\overline{)8}^{2}$ $2\overline{)8}^{4}$ $1\overline{)8}^{8}$ $\frac{1}{2}\overline{)8}^{?}$

Proof:
$1 \times 8 = 8$
$2 \times 4 = 8$
$4 \times 2 = 8$
$8 \times 1 = 8$
$16 \times 1/2 = 8$

Again, the same results are achieved when "Mary's Rule" is applied. Try as we will, we will never find an exception to "Mary's Rule."

It has not been proved! But the pupils will have a well-founded feeling that it is probably a true statement.

Moreover, they have become more able in dividing by "inverting the divisor and multiplying."

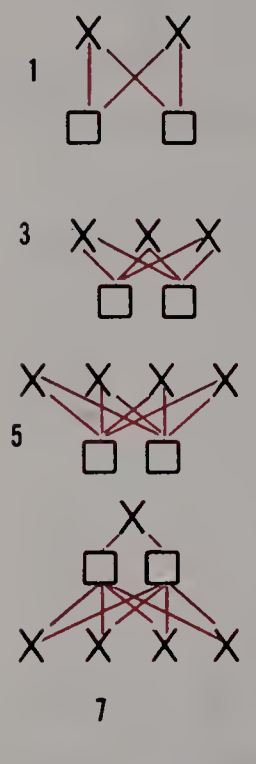
## V. TABLES AND NUMBER PAIRS

The difference between traditional and contemporary mathematics' curriculum content might be explained as the difference between a static and a dynamic viewpoint.

Traditional mathematics has been concerned with the memorization of facts and manipulations. "How much is $5 \times 3$? Answer: 15." The traditional approach can be characterized by the question, "What is *the answer?*"

Creative math on the other hand, is always asking the question, "What's going on here?" A typical situation that might be presented is:

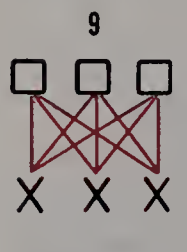
Draw a line from each □ to each × in each grouping and count the number of lines required.

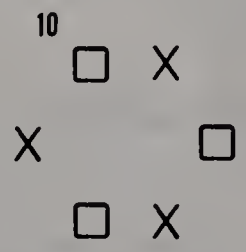



| | No. □ | No. X's | No. Lines |
|---|---|---|---|
| 1 | 2 | 2 | 4 |
| 2 | 2 | 2 | 4 |
| 3 | 2 | 3 | 6 |
| 4 | | | |
| 5 | | | |
| 6 | 2 | 4 | 8 |
| 7 | 2 | 5 | 10 |
| 8 | | | |

It is pretty clear from the above groupings that the arrangement or position of the figures doesn't affect the number of line segments. Only a change in the number of □'s or ×'s does. But, in each case, there are only two □'s.

Try 3 □'s and 3 ×'s.

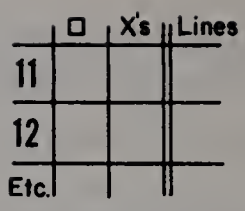


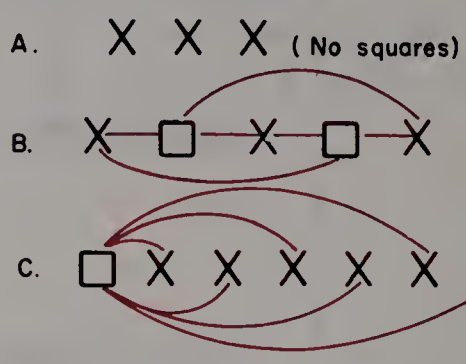

| | □ | X's | Lines |
|---|---|---|---|
| 9 | | | |
| 10 | | | |

These examples make clear that the number of lines is the same as the product of the number of □'s and the number of ×'s.

Now, get some individual investigation underway. Can any child find whether these two observations hold for all combinations and all arrangements? Keep score for every trial.




Some children will like to try tricky arrangements. How about:

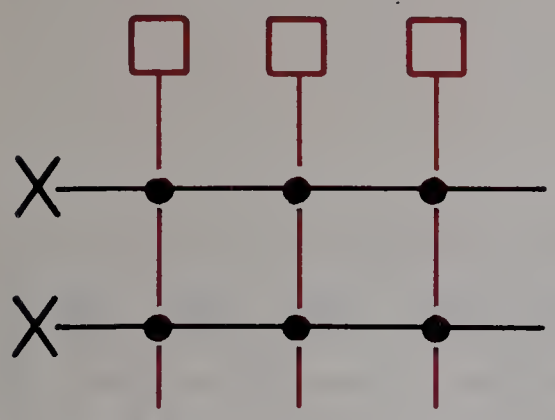



| | □ | X's | Lines |
|---|---|---|---|
| A | 0 | 3 | 0 |
| B | 2 | 3 | 6 |
| C | 1 | 6 | |

But no one will ever find a deviation from the patterns that begin to grow on the scoreboard. One way to add to a feeling of assurance that the patterns are consistent is to construct a scoreboard that exhausts one small area of possibilities. Obviously, if there are not at least one □ and one × there will be no lines. Also, if there is only one □, then there will be as many lines as ×'s—and vice versa. Moreover, it can be demonstrated that the position or arrangement has no influence.

| | □ | X | Lines |
|---|---|---|---|
| a | 2 | 2 | 4 |
| b | 2 | 3 | 6 |
| c | 2 | 4 | 8 |
| d | 2 | 5 | 10 |
| e | 3 | 3 | 9 |
| f | 3 | 4 | 12 |
| g | 3 | 5 | 15 |
| h | 4 | 4 | 16 |
| i | 4 | 5 | 20 |

The table above turns out to be nothing more unusual than a table of the facts of multiplication. Another picture can suggest this same basic operation. (The picture below is, in fact, a thinly disguised version of the first picture we used to begin building an understanding of this abstract idea.)



Unless someone keeps score of the pupil's results in this activity, the patterns will take long to emerge. And to further test the strength of these patterns, once established, another table can be employed, one that explores *all* the possibilities in at least one small area.

These scoreboards or tables are fundamental mathematical tools that help develop a real understanding of "what's going on?" No other technique is more widely used in the Math Workshop for Children materials. Similarly, no other technique is more widely used when the language of mathematics is used in scientific investigation, mechanics, and business. Almost every life situation involving mathematics can best be fully described by a table, chart, or graph—and every graph is a picture of a table.

Each child using the Math Workshop for Children materials is given the opportunity to build his own adding machine and multiplying machine beginning in the first grade.* These are the most fundamental tables of arithmetic—they include all the basic facts of arithmetic. These tables are also fruitful territory for pattern hunting. In the accompanying diagram are indicated some of the adding machine patterns—each of which can be written as a table by itself.

A. 4, 5, 6, 7, 8, 9, 10, 11, 12, 13—and every row or column reveals the same pattern.

B. 2, 4, 6, 8, 10, 12, 14, 16, 18—the even numbers.

C. 1, 3, 5, 7, 9, 11, 13, 15, 17—the odd numbers; and every parallel diagonal row contains all even numbers or all odd numbers alternately.

D. 9, 9, 9, 9, 9, 9, 9—and every diagonal parallel with this one contains a repetition of one number.

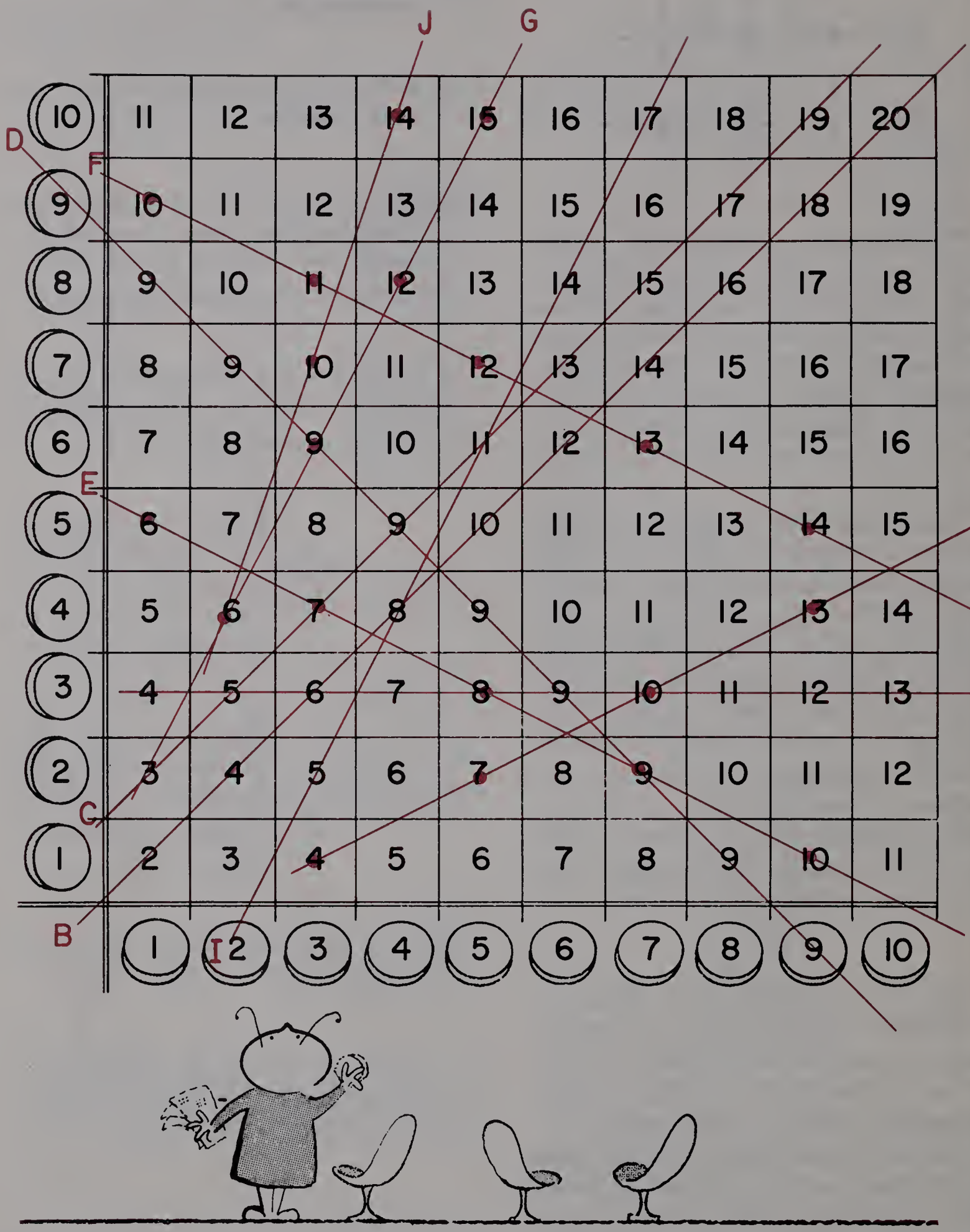
E. 6, 7, 8; 9, 10
F. 10, 11, 12, 13
} and every parallel line (down one and over two) reveals the same character of sequential numbers.

G. 3, 6, 9, 12
H. 4, 7, 10, 13
} Every parallel line (over one and up two) is a series with a common difference of 3.

I. 2, 5, 8, 11, 14—Same as G. H. (over one and up two).

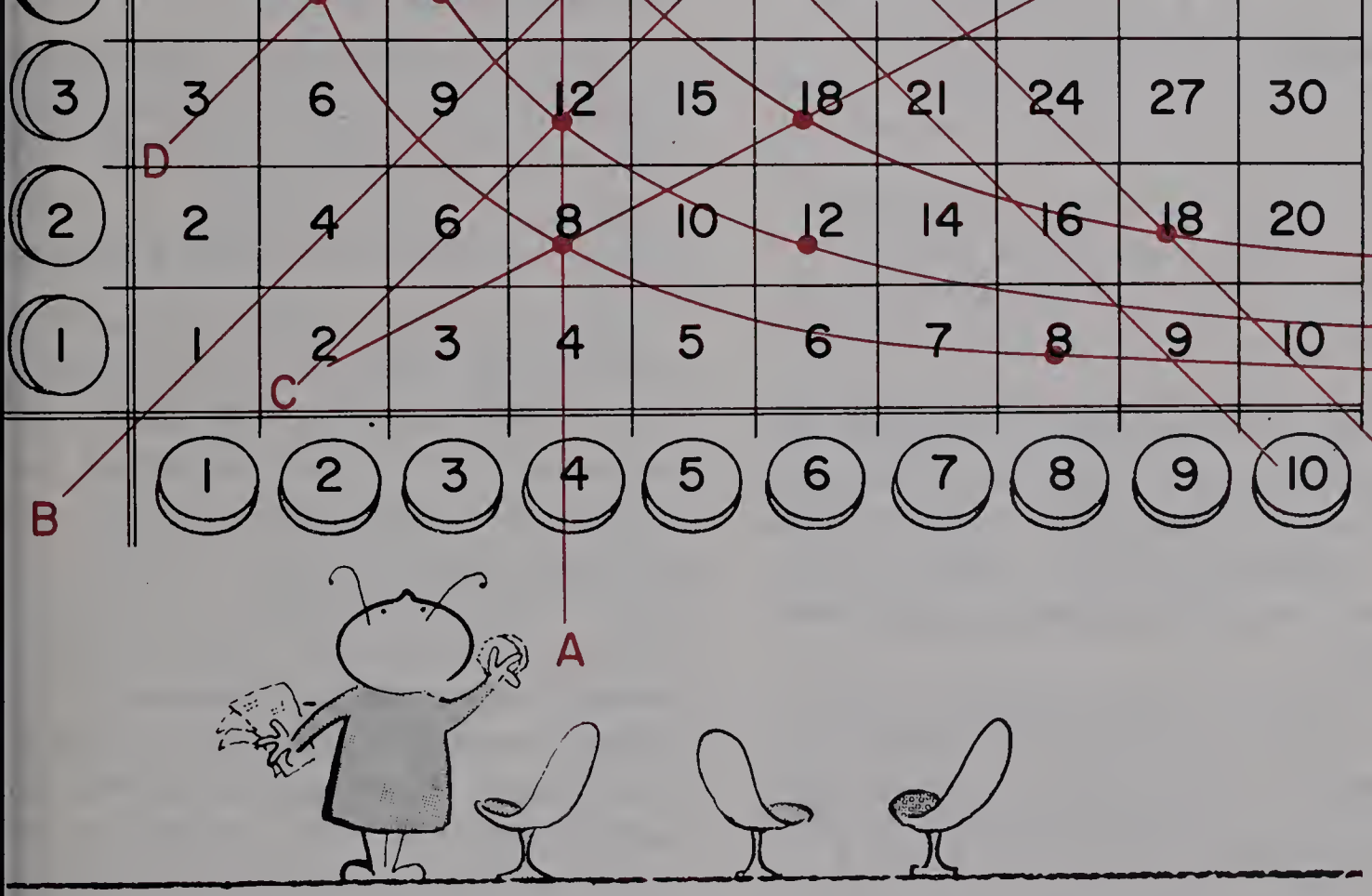
J. 6, 10, 14—and every parallel line is a series with a common difference of 4 (over one and up three).

*In these early experiences, pupils record the facts of addition in tables like the ones on the following pages.


J
G
D
F
E
A
C
B
10 11 12 13 14 15 16 17 18 19 20
9 10 11 12 13 14 15 16 17 18 19
8 9 10 11 12 13 14 15 16 17 18
7 8 9 10 11 12 13 14 15 16 17
6 7 8 9 10 11 12 13 14 15 16
5 6 7 8 9 10 11 12 13 14 15
4 5 6 7 8 9 10 11 12 13 14
3 4 5 6 7 8 9 10 11 12 13
2 3 4 5 6 7 8 9 10 11 12
1 2 3 4 5 6 7 8 9 10 11
1 2 3 4 5 6 7 8 9 10

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 10 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |
| 9 | 9 | 18 | 27 | 36 | 45 | 54 | 63 | 72 | 81 | 90 |
| 8 | 8 | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 | 80 |
| 7 | 7 | 14 | 21 | 28 | 35 | 42 | 49 | 56 | 63 | 70 |
| 6 | 6 | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 | 60 |
| 5 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 |
| 4 | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 | 40 |
| 3 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 |
| 2 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

F E G D C H I J B A

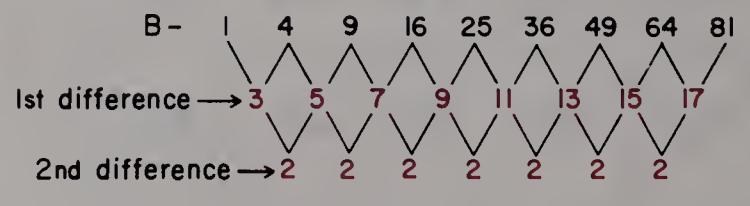


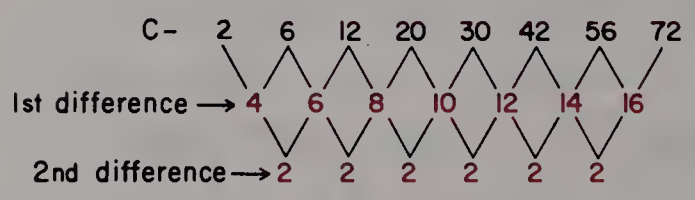
In fact, any line connecting any two numbers can be extended to include others and will always develop a pattern. (Up 2 and over 3 to the right will be a series with a common difference of 5; up 2 and over 3 to the left, a series with a common difference of 1; etc.)

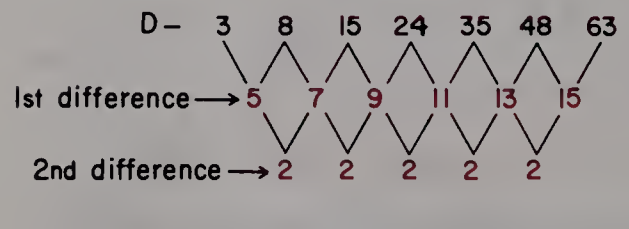
Now consider the *multiplying machine.* In the first place, each row and column is one of the *multiplication tables* as they used to be taught.

A- 4 8 12 16 20 24 28 32 36

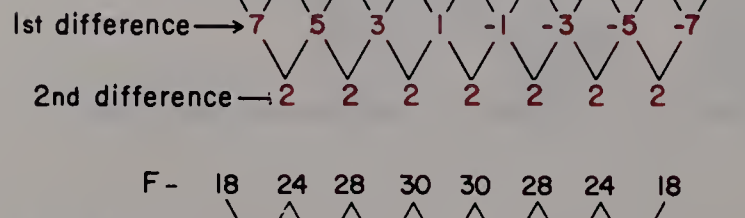
The similarity of the pattern in B, C and D suggest that something interesting is going on.

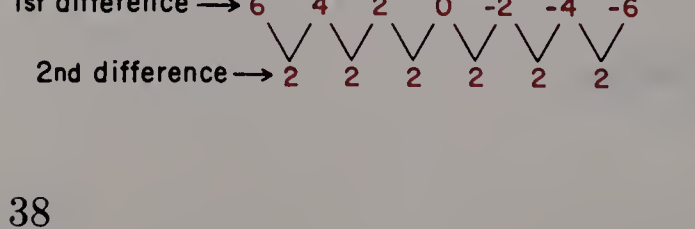


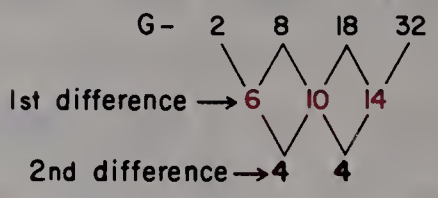




Every diagonal moving up and to the right generates a series in which each succeeding difference is two more than the previous one. The differences are either the odd numbers or the even numbers.




So, it becomes obvious that all diagonals perpendicular to the set examined above follow a very similar pattern except that the differences become smaller, converging to *1* or *0*, then increasing in reverse order.




These two devices are interlaced by pattern after pattern. Many are obvious and simple enough to detect by the newcomer to the world of numbers. And each pattern is a *table* when its elements are listed.

The fact that "tables are fundamental tools of mathematics" leads to a consideration of number pairs, since every collection of number pairs is a table, and every table can be considered a collection of number pairs.

Seldom in life situations is a number considered singly. And in mathematics, a number as a single entity has little interest. We use numbers in pairs, triples, quadruples, and larger groupings.

Can you think of a situation that deals with a single number? You could ask a child, "How many marbles have you?" He answers, "26." But look behind your question. You know that the number he has fluctuates from day to day, from game to game. You really want to know how many he has *at this time.* And if your concern were with his prowess at collecting, keeping, or winning, you would want to know how many he had the day before, and the day before that. And you would ask him tomorrow for his latest tally. To record the results you would need number *pairs* at least; one number to represent the day, the other to reveal the count on that day.

Matters of substantial concern are usually expressed in number pairs: 2 cans for 27¢; 72° at 4 P.M.; 35 miles in 60 minutes; the 4th day of the 7th month is Independence Day; sound travels at 1180 feet per 1/60th of a minute; Jim got 3 hits out of 4 times at bat; your chances are 50-50; or 5 for each 100; 200 yards with 1 drive; two bass and three pike.

In mathematics, after the numerals 0 through 9, we run out of any semblance of singles. In a very important sense, the numeral 27 is a pair, an *ordered pair*, the *first* numeral stands for *2 tens* and the *second* stands for *7 units*. The expression *3/4* is an *ordered* pair with several meanings: 3 out of 4, 3 divided by 4, in the ratio of 3 to 4, etc. And in everyday life it might mean March 4th.

In such everyday experiences it is possible to illustrate the difference between the *old* and the *new* approach to mathematics. Traditional textbooks are full of questions such as: "If candy bars are 4¢ each, how much would 3 bars cost?"

The *new math* is more interested in the larger problem the clerk faces. One customer will want 3 bars, the next 5, another only 1. The clerk's problem has a *table* for its solution.

| No. of Candy Bars | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| Total Cost @ 4¢ each | 4¢ | 8¢ | 12¢ | 16¢ | 20¢ | 24¢ | 28¢ |

Many daily experiences and almost all study of the world we live in (including scientific investigation) involve extensive use of *tables*—or *functions and relations*, as the mathematicians call them. Many teachers keep a month-to-month record of the height and weight of each child, his attendance, and other matters. Often this data is recorded in a table and then pictured by a graph of some kind.

Each of these activities affords an opportunity to involve every child in, and acquaint him with, a fundamental mathematic idea: the creation of a table. This powerful tool often helps him feel more surely that he knows "what's going on."

Because tables are so basic both to mathematics and to an understanding of many of the events in everyday life, we consider their creation and use as fundamental techniques in the Math Workshop for Children. These tables will enable pupils to see behind separate facts and situations to more general relationships. Tables can reveal patterns—patterns that mean movement. As we examine these patterns, we will continually be asking, "What's going on here?"

## VI. MEASUREMENT

A very sharp line must be drawn between *counting* and *measurement*—for measurement is *never exact.*

Measurement involves two variables: (1) The unit of measure, and (2) a comparison of the unit with the object being measured. Both of these variables are actually "approximations." All time spent on making these facts clear to pupils will be amply rewarded by a more thorough understanding of the problems involved in using the exact science of mathematics as the *language of measurement.*

"How many hands wide is your desk?" (The desks are, of course, not exactly the same width, but that fact is not mentioned.) Whose hand? Should the thumb be included? Should the hands be squeezed together? What if it doesn't come out even? . . . By questions such as these, the teacher can raise many of the sticky problems of measurement.

How many glasses in a quart? What size glass? How full? What about spilling?

The purpose of measurement activities in the beginning grades is solely to establish the ideas of (1) selection of a unit by agreement, and (2) observing and recording an approximate comparison.

How many "shoe lengths" wide is the room, stepped off toe-to-heel? Whose shoes? Perhaps several children can match their shoe lengths, and agree that Jerry's will be the unit because he and several others have shoes "approximately" the same length. This will enable others to check up on Jerry.

The next step in measurement is its use in "comparing" things that cannot be placed side-by-side. Which dimension of the room is biggest? Is the desk higher than it is wide? Are two first grade rooms about the same size? Are two doors about the same height? Who lives closest to school? Are all city blocks the same length?

In each instance, preparation for the activity is extremely important. "How are we going to do the job?" The discussion of this question necessarily involves selection of some appropriate unit. Reasons will be given for having several children try the same measurement independently and compare the results. Often the discussion will turn to comparing "parts of a unit." The purpose is to anticipate problems that will arise and to underscore the true meaning of measurement—and its hazards as an approximation.

Discussions of measurement and of fractions may often develop nicely together. Historically, the whole numbers of counting became insufficient to *measure* accurately enough—and it was the need for finer units of measure that mothered the invention of fractions.

Collect 16 half-pint milk cartons and work out as much of the following table as possible: first the red in terms of the black; then the dotted entries can be worked out.

| | | | | |
|---|---|---|---|---|
| gallons | | | | 1 |
| $\frac{1}{2}$ gallons | | | 1 | 2 |
| quarts | | 1 | 2 | 8 |
| pints | 1 | 2 | 4 | 4 |
| $\frac{1}{2}$ pints | 2 | 4 | 8 | 16 |

Now, how about the "holes" in the chart. Can "2 pints" be expressed as "□ quarts?" Yes, as "1/2 quart." Also the number of half pints can be doubled (to extend the pattern) so the chart becomes:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| gallons | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 | 2 | 4 | |
| $\frac{1}{2}$ gallon | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 | 2 | 4 | 8 | |
| quarts | $\frac{1}{2}$ | 1 | 2 | 4 | 8 | 16 | |
| pints | 1 | 2 | 4 | 8 | 16 | 32 | |
| $\frac{1}{2}$ pints | 2 | 4 | 8 | 16 | 32 | 64 | |

Now, stretch out the chart so that it lists each individual number of half-pints. Fill in the chart with the equivalent numbers of quarts, half-gallons, and gallons.

| | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| gallons | $\frac{1}{16}$ | $\frac{1}{8}$ | $\frac{3}{16}$ | $\frac{1}{4}$ | | | | | | | | | | | | 1 | | |
| $\frac{1}{2}$ gallons | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{3}{8}$ | $\frac{1}{2}$ | | | | 1 | | | | | | | | 2 | | |
| quarts | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{3}{4}$ | 1 | | | | 2 | | | | 3 | | | | 4 | | |
| pints | $\frac{1}{2}$ | 1 | $1\frac{1}{2}$ | 2 | | 3 | | 4 | | 5 | | 6 | | 7 | | 8 | | 10 |
| $\frac{1}{2}$ pints | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |

After the whole unit equivalents are xhausted, fractions must be introduced, ome of which are far from obvious. If hree half pints are put into a quart bottle, ow full will it be? Will three half pints t into one pint bottle? The pupils can ork out one situation after another, uessing what is going to develop, carry- g out experiments, and recording results.

Depending on the class, the teacher ay decide to work with only two or hree different measures, to carry out all he experiments individually, or to rely eavily on the "patterns" to anticipate sults. Finally, it is possible to carry the tivity a step further (again using as any rows—or sub-units of measure—as he class can handle adequately).

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| quarts | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| pints | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| $\frac{1}{2}$ pints | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| Amount Purchased Computed in 1/2 Pints | 1 | 2 | 3 | 4 | 5 | 6 | 7 |

The number across the bottom indicates he amount purchased if measured in 1/2 nts. How can each purchase be made sing the smallest number of cartons? ecord the number of each required— 0" if none. And, strangely enough, never more than one of each size required to ake the purchase in the fewest number cartons. (Moreover, a pattern develops at some may recognize from elsewhere mathematics—the pattern of the binary stem of numeration.)

This same kind of activity looks as follows when another series of units of measure is considered:

| | | | | | | |
|---|---|---|---|---|---|---|
| quarters | $\frac{1}{5}$ | $\frac{2}{5}$ | $\frac{3}{5}$ | $\frac{4}{5}$ | 1 | $1+\frac{1}{5}$ |
| dimes | $\frac{1}{2}$ | 1 | $1+\frac{1}{2}$ | 2 | $2+\frac{1}{2}$ | 3 |
| nickels | 1 | 2 | 3 | 4 | 5 | 6 |
| pennies | 5 | 10 | 15 | 20 | 25 | 30 |

Dropping the quarter:

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| dimes | $\frac{1}{10}$ | $\frac{2}{10}$ | $\frac{3}{10}$ | $\frac{4}{10}$ | $\frac{5}{10}$ | $\frac{6}{10}$ | $\frac{7}{10}$ | $\frac{8}{10}$ | $\frac{9}{10}$ | 1 | $\frac{11}{10}$ | $\frac{12}{10}$ |
| nickels | $\frac{1}{5}$ | $\frac{2}{5}$ | $\frac{3}{5}$ | $\frac{4}{5}$ | 1 | $\frac{6}{5}$ | $\frac{7}{5}$ | $\frac{8}{5}$ | $\frac{9}{5}$ | 2 | $\frac{11}{5}$ | $\frac{12}{5}$ |
| pennies | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |

Using fewest number of coins:

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10¢ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 5¢ | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| 1¢ | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 3 | 4 | 0 | 1 | 2 |
| Total | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |

Another series of tables can start out.

| | | | | |
|---|---|---|---|---|
| yards | $\frac{1}{3}$ | $\frac{2}{3}$ | 1 | $1+\frac{1}{3}$ |
| feet | 1 | 2 | 3 | 4 |
| inches | 12 | 24 | 36 | 48 |

In fact, every table of weights and measures can be handled in similar fashion. This kind of activity opens the way for interesting discussions, experiments, and opportunities to record results. These experiences will help pupils react automatically to many of the facts of arithmetic and will reinforce those that are marginal.

## VII. PROBLEM SOLVING

The Math Workshop for Children has charted a three-point course to equip pupils with better mathematical tools:

**A.** *Mathematical Headlines.* Pupils learn to look for the conditions in the problem, rather than the "answer."

**B.** *Tables.* Pupils learn to search in tables and graphs for relationships among a number of facts found in everyday experience.

**C.** *Pure Mathematical Problems.* Pupils learn to explore pure mathematical problems that "go somewhere."

### A. Mathematical Headlines

Mathematics is a central part of the elementary curriculum because it has so many applications or analogies in the world of events. If a teacher goes to the trouble of teaching a child that:

$$2+3=5$$

he hopes that this fact can be useful in the child's everyday life as well as valid mathematically.

Once a child knows that $2+3=5$, look for situations in which he needs to know that $2+3=5$ . . . at the store . . . setting the table . . . keeping score in a game . . . satellites in orbit . . . telling time.

Each child should then create a story problem of his own in the means of communication he prefers—telling the story, writing it, drawing it in pictures tha illustrate it, manipulating objects tha demonstrate it.

By encouraging pupils to make up the own story problems, the teacher takes th usual emphasis off the answer and make the situation turn on the mathematica statement—the *mathematical headline* c title for which the children create the own applications. The answer-oriente approach distracts children, and it is n really the *answer* that is most desired— is the mathematical statement.

A traditional textbook might phrase problem as follows: "John paid 2¢ f licorice and 3¢ for gum. How much did h spend?" The correct answer is 5¢. If th matter ends there, the heart of th problem situation has been ignored. Th teacher can resort to one means of rescu by asking, "Why is that true?" "Becau 2 cents plus 3 cents equals 5 cents." "Wh is that true?" "Because $2+3=5$!"

If the child doesn't know already tha $2+3=5$, he should not have been face with that problem. If his attention centered on an answer that he alread knows, the problem becomes utter trivial. The teacher's first and almost on concern is that this problem illustrate th relationship between mathematics an the world of events—a relationship th emerges more and more clearly as t teacher continues to bring the mathe matical statement always to the forefro

After children become familiar with th creation of story problems appropriate a given mathematical title or headlin they should be given opportunities match given titles to a given picture.

| PICTURES | Complete all sentences. Check the one suggested by the picture. |
|---|---|
| X X X<br>X X | 5 + 2 = ◯<br>3 + 2 = □<br>2 + ⬡ = 1<br>◇ - 6 = 2 |

Which title is correct? . . . Why 3+2 =5? . . . Why not each of the others? . . . What do these other titles say? Could he picture suggest another headline bout as well as 3+2=5? . . . In what ay does it suggest the other members of he basic fact team to which 3+2=5 elongs: 2+3=5; 5−3=2; 5−2=3? Once he children have grasped this story prob- m, they can be presented with another:

"Jack bought 5 oranges and 4 apples. How many pieces of fruit did he buy?"

The main concern, remember, is not ith the correct answer. Rather, the ajor concern is with the mathematical tatement that presents the conditions of he problem. It is:

5+4=□

After the statement has been written spoken, and completed with "9", there one matter left which has to be dealt ith. The answer to the mathematical tatement is not the answer to the prob- m. Lead the children back to the roblem to find in its context an interpre- tion for "9." Why is it not "9 apples" "9 oranges?" Because Jack bought me of each. It's "9 pieces of fruit"— nd the clue for the answer is in the nguage of the problem itself.

Further interest can be created by posing such questions as: "If we changed the story, how would we have to change the headline? If we changed the headline, what changes would be needed in the story?"

Many children will not be able to create thoughtful, imaginative, interesting, and significant problems at the outset.

Some of their "story problems" will be ambiguous or questionable interpretations of the headline. These become excellent material for class discussion. Some of the story problems will be short, others long, some funny, some hilarious, and some no more interesting than those found in typical text books. Like any other creative activity, a dynamic usually develops that challenges every child to his utmost. Everyone is challenged; everyone can succeed!

As in any other classroom activity, the teacher must play a vital role as a thought starter. The approach to mathematics of the Math Workshop for Children assists the teacher by lifting the road blocks to creativity usually built into "word problems."

Most important, the whole approach to problem-solving begins with and depends upon discovery of the relationship that exists between mathematics and life situations. As soon as the child gets the sense of what's going on in the mathematical headline, he's on the way to solving the problem in the familiar surroundings of everyday language.

The Math Workshop for Children has designed several forms for use in this activity. An example of one form follows, showing three different ways of using the form:

**A**

$13 - \square = 5$

Story: _____

_____

_____

_____

_____

_____

_____

Scratch pad:

**B**

Story: How many days are there in 3 weeks ?

_____

_____

_____

_____

_____

Scratch pad:

**C**

Story: _____

_____

_____

_____

_____

_____

_____

Scratch pad:

In A, the headline is given, and the jo is to create a story. In B, the headline i left out, and the story given. In eithe case the child can use the "scratch pad" for illustrating the story or to perform an computations necessary to complete th mathematical statement. For example, i one activity he may be required to con vert 13 feet into inches.

In C, a sketch is drawn on the "scratc pad," and the child's ability to create i really turned loose. Here he must writ the headline and the story to fit th sketch.

Another form in the Math Worksho for Children is pictured below:

| PICTURES | Complete all sentences. Check the one suggested by the picture. | Fill in all of the basic fact team suggested by the picture. |
|---|---|---|
| | $5 + 2 = \boxed{7}$ | $3 + 2 = 5$ |
| | $\hexagon{6} - 2 = 4$ | $2 + 3 = 5$ |
| | $3 + \hexagon{2} = 5$ | $5 - 3 = 2$ |
| | $6 = \hexagon{8} - 2$ | $5 - 2 = 3$ |

There are at least four jobs to be completed on such a page:

Job 1—Make complete and true statements of the four sentences.

Job 2—Select the one that would be the most appropriate caption for the picture.

Job 3—Draw another picture that could carry the same title.

Job 4—Write the four members of the act team, any one of which would be a true statement for either picture.

Children may also be asked to make up short story about either picture. Here are some examples:

"There are three squares and two circles. There are five shapes in all."

"After Alice drew two circles, she drew three squares. If these were rings, she would have one for each finger on her hand."

"There are five shapes, only two of which are circles. The rest are squares. There are 12 sides to the three squares."

"I had a nickel and spent 2¢. I got 3¢ change and I didn't lose any. How much have I left?"

As the story problems move up the scale of difficulty, the teacher should continue to stress the central role of the mathematical statement—the fact that this statement is what the pupils are really after. The teacher should continue to call upon children to create stories that are consistent with a mathematical headline.

A most dramatic test of the effectiveness of interpreting problems as mathematical statements can easily be made by any teacher who encounters trouble with word-problems involving "per cent." Such a problem might be stated as follows:

10% x 50 = □ 10% x □ = 5 □% x 50 = 5

(We are assuming, of course, that children have advanced to the point where they know how to find the numbers needed to make each of these headlines into a complete statement. If there is any hesitation in finding such numbers, any attempt at word-problems would be doomed to failure.)

The headlines, then, are:

10% x 50 = [5] 10% x [50] = 5 [10]% x 50 = 5

Here are those infamous "three cases of per cent"—if we agree that the box tells us the part of the statement that is missing at first.

The problem is now simply that of translating a mathematical headline into a life situation. Any situation involving the idea of "per cent" can be used as an example.

In working from headlines to stories and the reverse, pupils will reveal the individual difficulties which they face. If a pupil cannot complete the headlines, then he doesn't know the basic facts of mathematics. If he cannot create a story to fit a headline, then he has not grasped the idea of the mathematical statement.

As they gain experience in translating mathematics into life situations, pupils will have less trouble in writing the mathematical statements or formulas to express the conditions of a story problem.

### B. Tables

Throughout this book, it has been stressed that tables are excellent means of

revealing relationships in mathematics. In addition, a guiding principle in the Math Workshop for Children is that problems should revolve around real life situations. These situations are often presented in tables or graphs which combine many details into manageable forms.

For example: The Math Workshop for Children reprints a grocery store's newspaper ad along with Mrs. Clarke's shopping list. Some items on her list may be clothing. Some of the food items on her list may not be mentioned in the ad. What light does this ad throw on her problem? Which grocery items on her list are included in the ad? Which are not? Mrs. Clarke's list includes 2 cans of tomato soup, and the ad announces a special of three 15¢ cans for 35¢. What will Mrs. Clarke probably do when she gets to the soup section? How much will individual items cost? About how much will her final bill amount to? What about the unlisted items? Children might ask their parents about prices for these—and the children can compare the results the report.

Another experience revolves around table showing the date each membe nation was admitted to the Unite Nations—and the population of eac nation. The goal of the discussion range from the elementary experience of readin a separate piece of data from this tabl to comparisons between populations, t the (perhaps disturbing) fact that eac country has one vote regardless of its size

A series of observations from the are of science can be the basis for an almos completely open-end activity. On example might be the temperature of pan of water on a stove taken at one minute intervals as it rises from roo temperature to the boiling point. Mo advanced children can be challenged t find a rule that will take them from o reading to the next.

A cook book recipe for cake icing included. The amount of icing is sufficie for the tops and sides of two 9-inch layer

| | Brown Sugar | Water | Egg Whites | Vanilla | Chopped Nuts | Salt |
|---|---|---|---|---|---|---|
| Tops and sides of two 9″ layers..... | 3 cups | 1 cup | 2 | 1 teaspoon | 1 cup | 1 pinch |
| Tops and sides of four 9″ layers.... | | | | | | |
| Tops and sides of six 9″ layers..... | | | | | | |

First, the recipe is rewritten in the above tabular form. Then the recipe is doubled for four 9-inch layers, and then tripled. Why are the amounts in the bottom row the sum of the first two rows? How much is a pinch? Suppose what was needed was enough for the top and sides of a 3-layer cake? How full should t cups and teaspoons be?

Many factual situations can be mo fully understood when they are presente as tables. Transportation schedules a published as time-tables. Interest tabl

re provided for bank clerks to eliminate difficult computations that might introduce error. Most books have an index. Height—weight relationships are organized in tables.

The grocery ad is a table. Mrs. Clarke's shopping list is another. A recipe is a table. The observations of a scientist are usually recorded in tables. When a person going to the circus purchases a ticket at the box office for $1.50, the cashier must have a table in her head, displayed in front of her, or be able to construct one quickly:

1 ticket for $1.50
2 tickets for $3.00
3 tickets for $4.50

Almost every story problem in a traditional textbook is an isolated fragment torn out of a structure. The Math Workshop for Children focuses on the structure. This integrated approach to problem solving provides teachers with opportunities for coordinated activities that are impossible in unrelated story problems. And almost always the general discussion of problem situations will uncover by-ways that lead out in interesting directions for the more able children.

### Pure Mathematical Problems

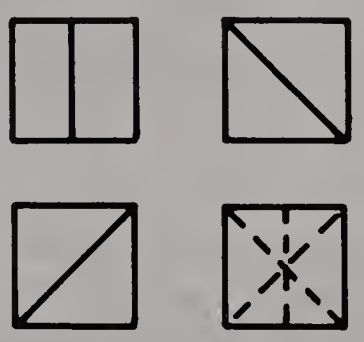
1. John had 9 marbles and lost 5. How many did he have left?
2. Mary bought a pencil for 4¢ and an eraser for 3¢. How much did she spend?
3. Mr. Brown had 13 cows. Eleven were black. The rest were white. How many were white cows?

Two, of course! But of those two, how many had brown spots? Can anyone determine exactly? No, but it is possible to make an observation, namely: that there are only three possible answers! (1) None had brown spots; (2) one had brown spots, and (3) two had brown spots. The response to this question takes much more thought than the answer to problem number "3."

The teaching of mathematics offers unlimited opportunities to ask questions requiring the highest kind of reasoning without going outside the world of numbers. Cows, pencils and marbles are not even required. Let's take 10 trips into this land of pure mathematical problem solving.

*Trip 1.*

How many ways are there to cut a square of paper into halves?



Do all these cuts pass through the center of the square?

Is there any straight cut through the center that does not divide the square in half? Is the same true of a rectangle? A circle? A triangle? An irregular shape?

*Trip 2.*

After a piece of paper has been cut in half, continue the process—cut a half into two halves, and then again. How many such cuts can be made?

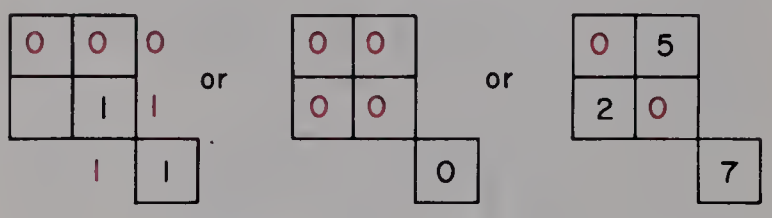
*Trip 3.*

Equilateral triangles, squares, regular pentagons, hexagons, and similar geometric shapes all have sides of equal length. Can we make regular polygons with as many sides as we please?

As the number of sides increases, what shape does the figure more nearly approximate? Can a regular shape exist with less than three sides?

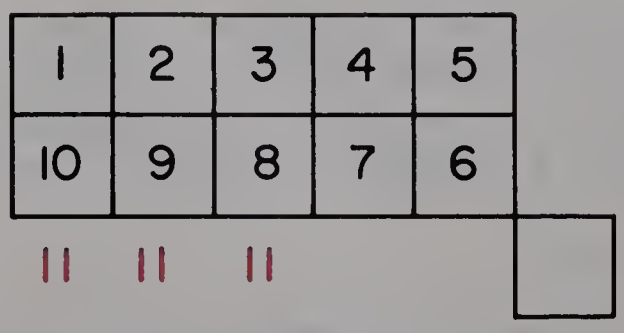
*Trip 4.*

Does the cross number puzzle always work for the addition of whole numbers? Fractions? Decimals? Does it always work for multiplication? If so, why? Does it work if there are more than four basic frames? Would it work with six? Eight? Nine? Is there a way in which it will work with five, seven, or eleven frames? What might be said about an addition cross number puzzle that looks like this:

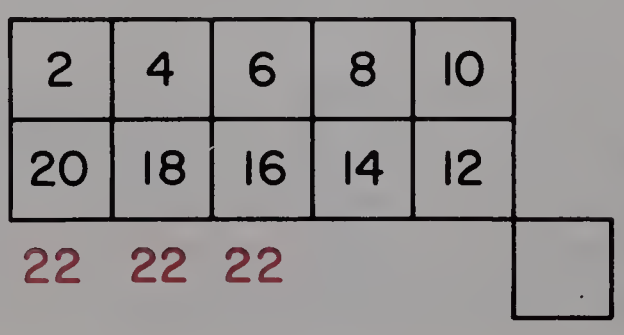



*Trip 5.*

What is the sum of all the whole numbers from 1 through 10? Is there a short-cut for finding this sum? Is a cross number puzzle in the following form a clue:




How about the sum of the first 10 even numbers? Is a cross number puzzle in the following form a useful clue:




Could you expand this technique as a short-cut to finding the sum of the first 15, 20, or 100 numbers? Suppose you tackle the sum of the next 10 whole numbers greater than 2 (3 to 12 inclusive).

*Trip 6.*

In how many ways could a couple (a boy and a girl) be selected from a group consisting of Bob, Jim, Helen and Sarah? How many more couples would be possible if Dick joined the group? If Dick and Jane both joined the group? Record the results.

| GROUP | No. of Boys | No. of Girls | No. of Couples |
|---|---|---|---|
| a. Bob, Jim, Helen, Sarah | 2 | 2 | 4 |
| b. Same with Jane | 2 | 3 | |
| c. Same with Dick and Jane | 3 | 3 | |
| d. Same without Jane | 3 | 2 | |
| e. Same as "d" with Ned | 4 | 2 | |

(The results suggest that we could define multiplication in terms of, "How many couples . . . ?")

*Trip 7.*

Numbers in the series 0, 2, 4, 6, 8 . . . are called "even numbers." All others are called "odd numbers." For convenience we will use "E" and "O" for even and odd.

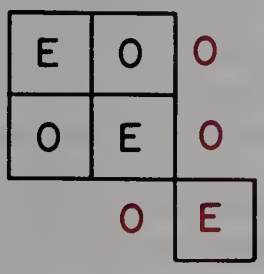
E+E=? (This statement asks whether the result will be even or odd if we

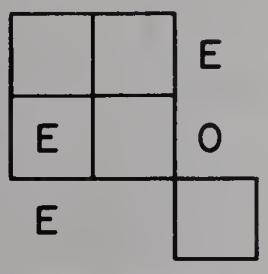
add two even numbers.
Answer: E)

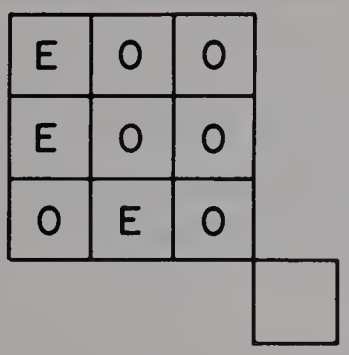
| | |
|---|---|
| O+E=__ | E−E=__ |
| E+O=__ | E−O=__ |
| O+O=__ | O−E=__ |
| | O−E=__ |
| E×E=__ | E÷E=__ |
| O×E=__ | O÷E=__ |
| E×O=__ | E÷O=__ |
| O×O=__ | O÷O=__ |

Cross number puzzles can be employed using these generalized numbers:

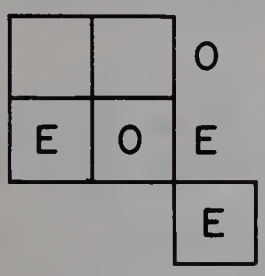
**Using Sums:**

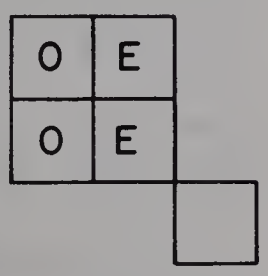


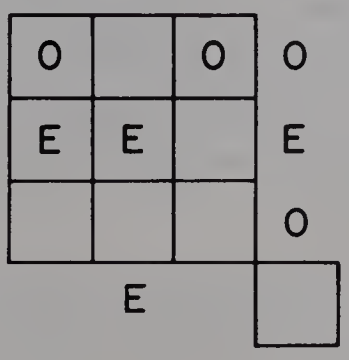




**Using Products:**




(And lots more! E×(E+O)=__, for example.)

*Trip 8.*

"Think of a number" is a game that can be introduced at any level, 1st through 8th grade.

Select a number from the group below. How many "yes" or "no" questions will have to be answered to determine the selection?

1, 2, 3, 4, 5, 6, 7, 8

The correct answer is, "Only three questions." Several sets of questions will do the job, provided each question divides the remaining group into two parts. For example:

Is the number greater than 4?

Is it an even number?

Devise other questions that divide the set of numbers above into two parts.

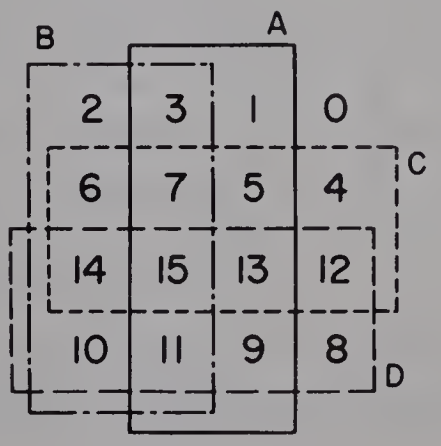
How large may the set of numbers be if four questions are permitted?

In primary grades, the situation might be presented in the following fashion. Think of a letter in the following set:

A a B b

Questions which divide the set into two parts are: (1) Is it a large letter? (2) Is it "bee"? (3) Is it on the end?

In the sixth grade, the following diagram might be used:



Is it in A? Is it in B? Is it in C? Is it in D? The answers to these four questions will pinpoint the number.

*Trip 9.*

In "Quietland" no one can talk because noise causes earthquakes. Consequently, the inhabitants have to count on their fingers. Under the impetus of need, they learned to count to 31 on the fingers of one hand—and to 1023 using both hands. How?

Well, they used to paint dots on their finger nails as shown below: (Of course, now they don't need these reminders.)

If the little finger is up alone, that's "one." If the ring finger alone, that's 2. If both are up, 2+1 or 3. If only the middle finger, that's 4. What would "5" be? Middle finger and little finger.

Here is how they count to 31 on one hand. In the chart a "1" means that finger is up and "0" means that finger is down.

| Finger | | | | | |
|---|---|---|---|---|---|
| Thumb | Fore | Middle | Ring | Little | |
| 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 0 | 0 | 1 | 0 | 2 |
| 0 | 0 | 0 | 1 | 1 | 3 |
| 0 | 0 | 1 | 0 | 0 | 4 |

And in "Quietland," they have only pennies, nickels, and quarters for money. How can they make up all the different amounts with the fewest coins?

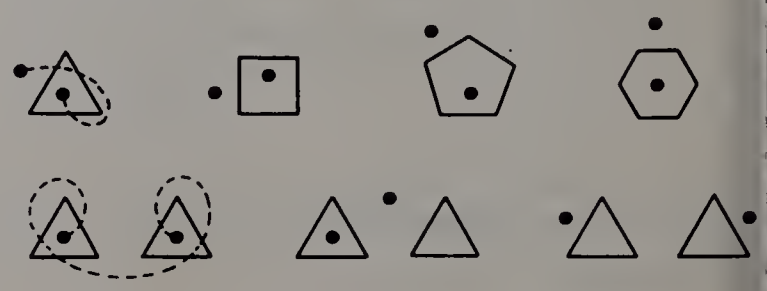
| No. of 25¢ | No. of 5¢ | No. of 1¢ | Our Money |
|---|---|---|---|
| 0 | 0 | 1 | 1 |
| 0 | 0 | 2 | 2 |
| 0 | 0 | 3 | 3 |
| 0 | 0 | 4 | 4 |
| 0 | 1 | 0 | 5 |
| 0 | 1 | 1 | 6 |
| 0 | 1 | 2 | 7 |
| 0 | 1 | 3 | 8 |
| 0 | 1 | 4 | 9 |
| 0 | 2 | 0 | 10 |
| 0 | 2 | 1 | 11 |

This chart should not be carried past 124¢ in our money unless "Quietland's" 125¢ bill is introduced. Why?

*Trip 10.*

How many of the following shapes can be drawn without raising the pencil or retracing a line?

Or, a very similar question can be asked in another way. In the following pictures, try to connect the two dots with a continuous line that crosses each side of the polygon or polygons at least once but only once.



In how many instances is it possible?

Mathematics is full of such delightful problems.

None of these problems is comparable to the usual mathematical trick. Each is based on a fundamental principle of mathematics which, once understood, opens up unlimited vistas with pathways leading everywhere throughout the structure of mathematics. Each problem is an introduction to a truly open-end area of future study.

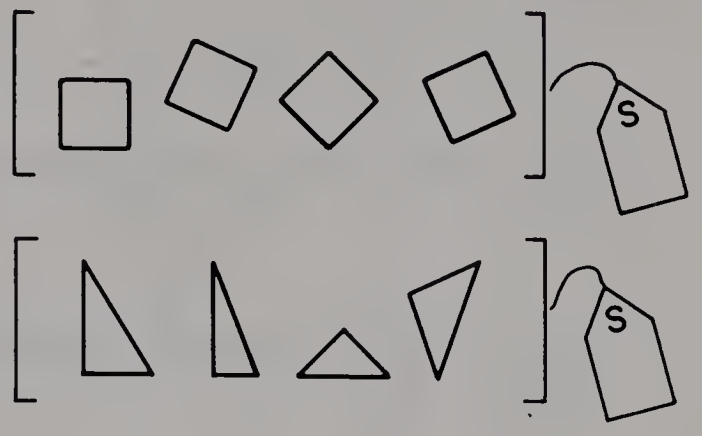
## VIII. GEOMETRY

Every child has intuitive notions about the space he lives in. While his understanding is vague, he has ideas about *between, points, lines, crosses, doesn't cross, up, down, meet, corner, far, near, inside, outside, on.* He knows that two objects cannot occupy the *same space* at the *same time.* He moves himself from one place to another. He looks around and finds things in different places, and perhaps rearranges them.

Early ideas of space grow from the recognition of *shapes* and *distances*, or the spatial relationship between shapes. In each experience, in trying to *explain* these shapes and relationships, the child is developing a vocabulary of *abstractions,* just as he does with *numbers.* He notices similarities and differences—and uses words to describe them. He compares the sizes of objects and distances; he compares the outline or contour of objects: flat, not flat; round, square; like a ball, like a block.

The primitive drawings made by children tell us much about what they see. These drawings give us clues as to the elements chosen by the children to differentiate objects. We can get an idea of the differences by which each child distinguishes people and cows, houses and stores, cars and space ships, friends and enemies, adults and children.

Geometry is based on those abstract ideas we agree upon as most useful and fundamental in describing the space we live in. It is the *structure* of space as we empirically find it. In the beginning then, we are concerned about developing a descriptive vocabulary. What is similar in each of the following collections of shapes:

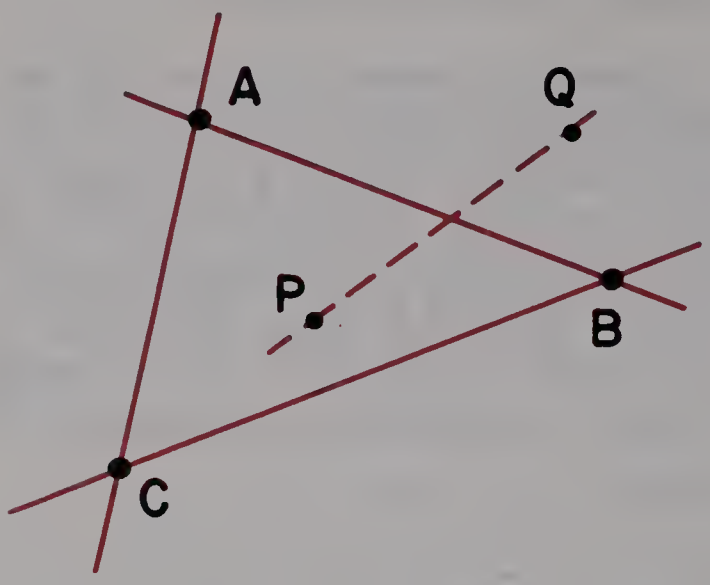



The Math Workshop for Children repeatedly employs basic shapes, beginning with the earliest counting experience. Instead of relying solely on sailboats, drums, dolls, horses, and cows, it employs circles, squares, triangles, hexagons, ovals, and other geometric shapes. In pictures used to suggest the facts of arithmetic will be found comparative areas of many geometric shapes, lengths of straight lines, arcs, and irregular curves.

Actual work in geometry in the first year is limited to using a pencil and straight-edge, and perhaps a compass. The main goal is to develop the ideas of point, lines, points on a line, connecting two points with a line, drawing a long line through two points, drawing lines that cross or do not cross.

The method employed by the Math Workshop for Children involves: (1)

following directions, and (2) talking about what happens. For example:



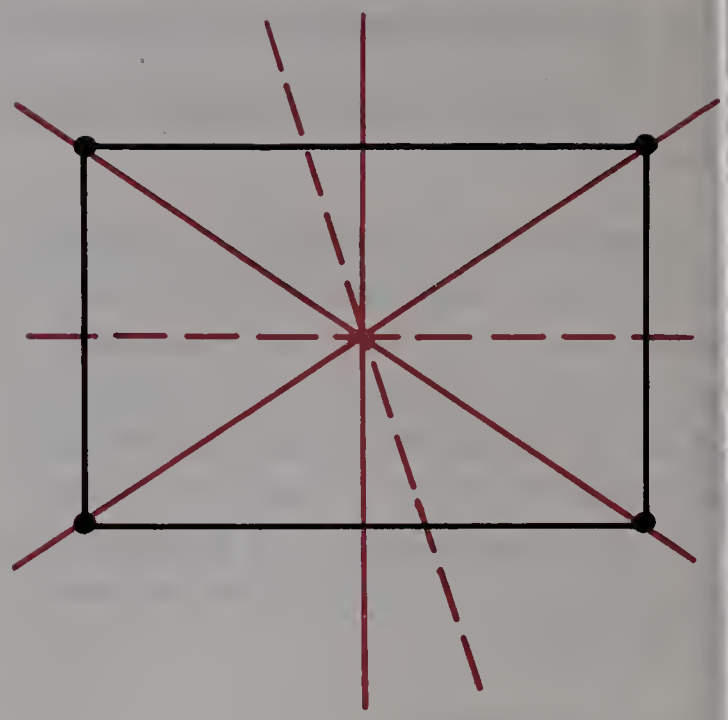

1. Draw a long line through points A & B.

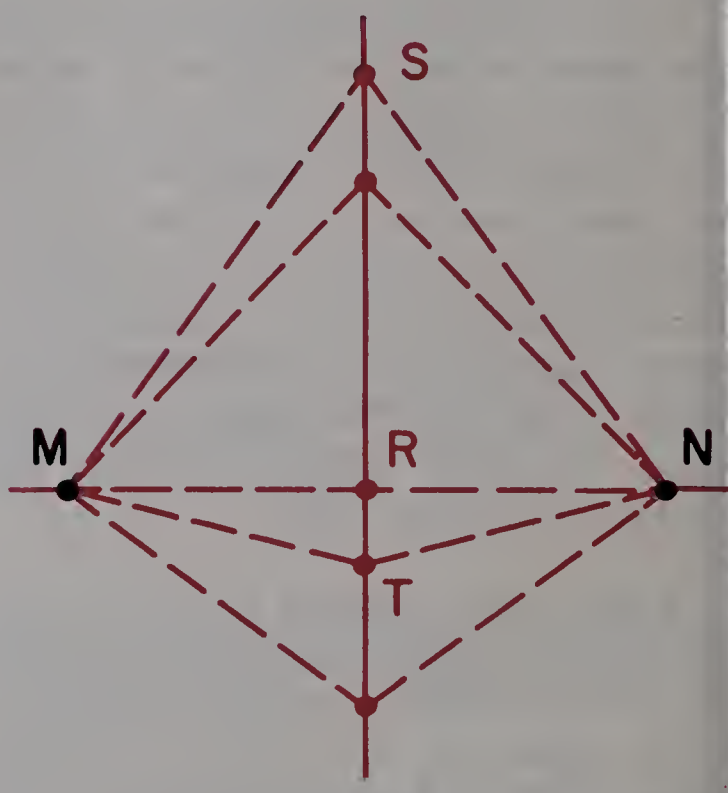
2. Draw a long line through points B & C.

3. Draw a long line through points C & A.

What happened? There are three intersections, three places where a pair of lines cross. Every line crosses the other two. Line segments AB, BC, and AC form a triangle. Point P is inside the triangle; point Q is outside the triangle. Line segment BC is longer than line segment AC. Line segment AB is the same as line segment BA. What would happen if more lines were drawn, connecting point A, B, and C to point P? What happens if point P is connected with point Q?

Different experiences grow out of drawing a line that divides a rectangle or square into parts of approximately the same size.





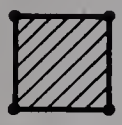

The problem stated here is to find a point that is about the same distance from M and N, such as R. Are there other points the same distance from M and N? Yes: points S and T. And a few more points would suggest that there are many points—lying on a long straight line through points S, R, and T.

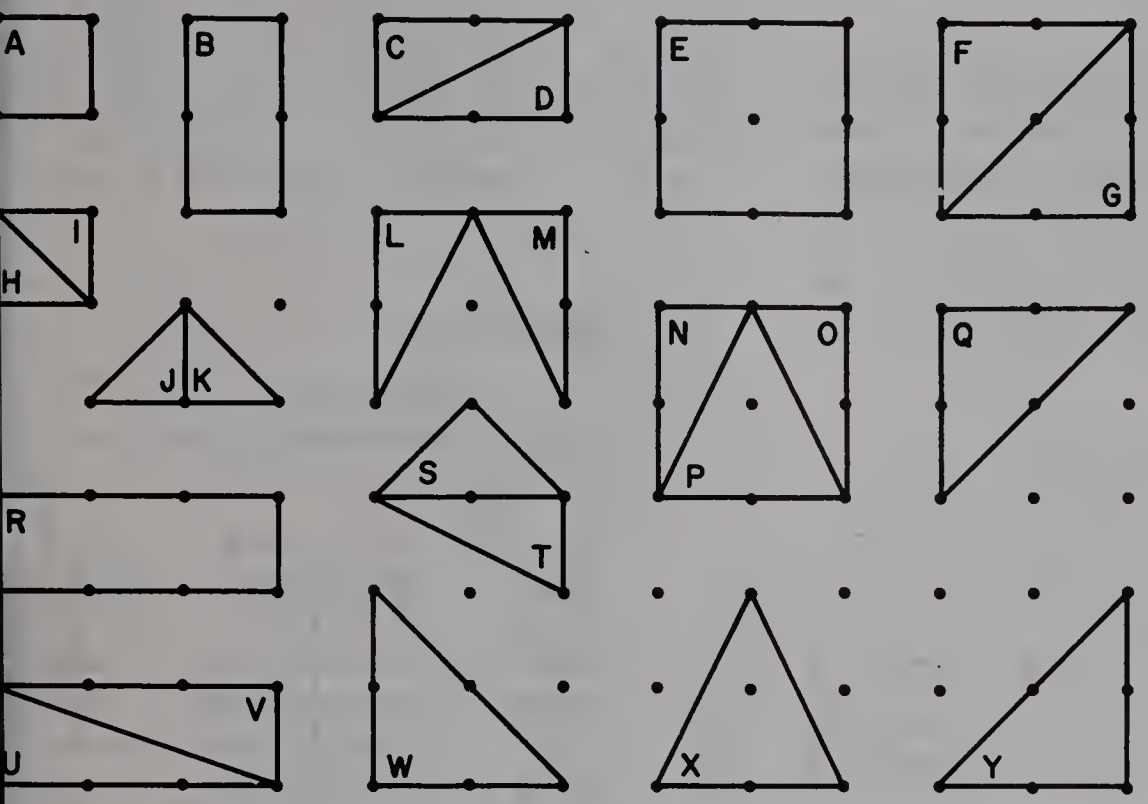
Intuitive ideas about properties of geometric figures can be stimulated by challenging exercises. The following page from a Work-Text, shown here in reduction, is almost completely self-explanatory

to most third grade children as soon as the idea of "unit size" has been discussed and illustrated.

Children work through this page, recording their results on the "scoreboard."

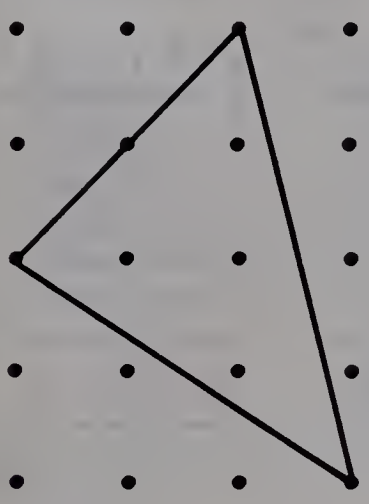
As they do, they are being led by small steps toward an important generalization, namely, that the area of any triangle is half the product of any side as a base and the measure of the perpendicular from the other vertex to that side—extended, if necessary.

If this size  represents 1 unit, what do the other shapes represent. Keep a record.




| | Units represented | | Units represented |
|---|---|---|---|
| A | 1 | N | |
| B | 2 | O | |
| C | 1 | P | |
| D | 1 | Q | |
| E | 4 | R | |
| F | 2 | S | |
| G | 2 | T | |
| H | $\frac{1}{2}$ | U | |
| I | $\frac{1}{2}$ | V | |
| J | $\frac{1}{2}$ | W | |
| K | | X | |
| L | | Y | |
| M | | | |

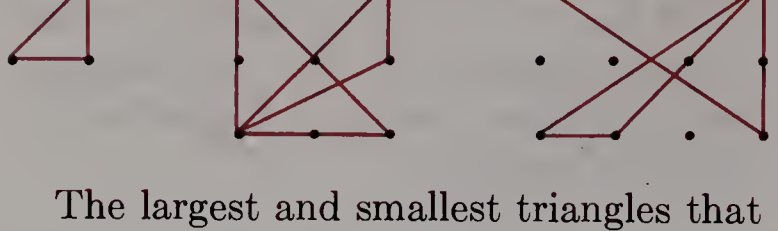
All polygons can be triangulated—as the Egyptians well knew. A developing concept of the area and other properties of a triangle can be helped along by attacking the problem from many directions.



The area of any triangle such as the one above can be found by inscribing it in a rectangle. The area of the triangle is the rectangle less the sum of those triangles in the rectangle outside the subject triangle. Thus, the area of any triangle constructed by interconnecting any three points in a unit grid can be completely determined.

Many interesting side-trips can help deepen the understanding of a triangle so constructed. What are the largest and smallest triangles that can be drawn in the following grids—with at least one vertex touching each outside row or

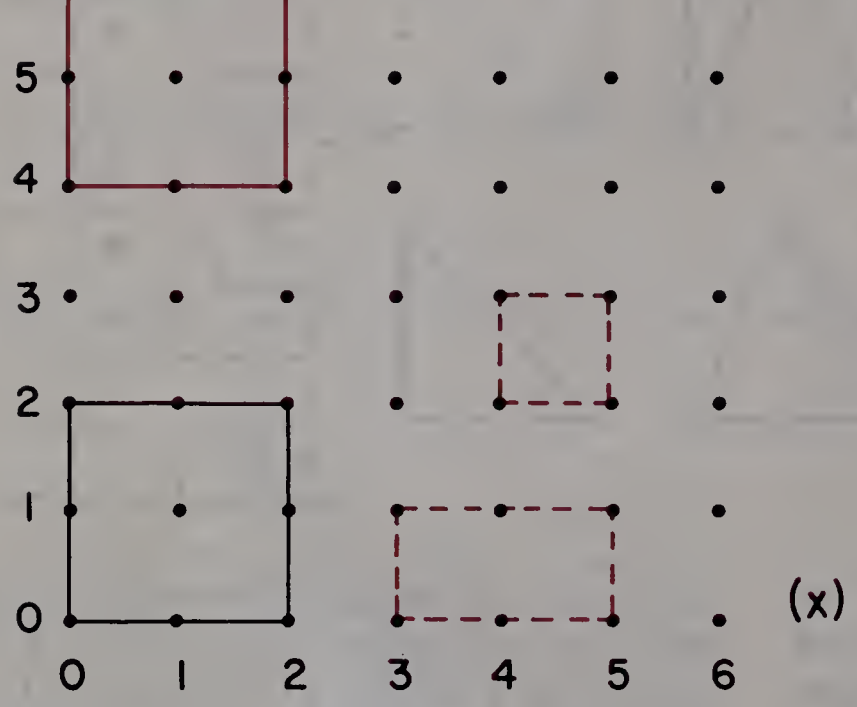
column? How many different shapes can be drawn? How many of different areas?



The largest and smallest triangles that can be drawn on the above grids have been indicated, but the attempts to prove that this is true leads to revealing discussion and arguments.

Expanded grids become playgrounds for making shapes of all descriptions. They move around, turn upside down, grow, shrink, become tall and squat. And, all the time, these geometric forms become quite familiar subjects for contemplation.

Each point on a grid can be named by a number pair. The first member of the pair denotes how far to the right of the point is, and the second how far above the "x" column it is. Thus, the bottom square in the grid below can be described by its 4 corners, or vertices, as: (0,0), (2,0), (2,2), and (0,2). By setting up various "rules" naming other vertices, we can move the square around the grid. Sometimes the square is distorted by the rule into something that is not a square.




Rule A:—

(x,y) goes to (x+1, y+4)

| x | y |
|---|---|
| 0 | 0 |
| 0 | 2 |
| 2 | 2 |
| 2 | 0 |

| x+1 | y+4 |
|---|---|
| 0+1 | 0+4 |
| 0+1 | 2+4 |
| 2+1 | 2+4 |
| 2+1 | 0+4 |

| | |
|---|---|
| 1 | 4 |
| 1 | 6 |
| 3 | 6 |
| 3 | 4 |

Rule C:—

(x,y) ⟶ (x ÷ 2 − 4, y ÷ 2 + 2)

| x | y |
|---|---|
| 0 | 0 |
| 0 | 2 |
| 2 | 2 |
| 2 | 0 |

| (x÷2)+4 | (y÷2)+2 |
|---|---|
| (0÷2)+4 | (0÷2)+2 |
| (0÷2)+4 | (2÷2)+2 |
| (2÷2)+4 | (2÷2)+2 |
| (2÷2)+4 | (0÷2)+2 |

| | |
|---|---|
| 4 | 2 |
| 4 | 3 |
| 5 | 3 |
| 5 | 2 |

Rule B:—

(x,y) ⟶ (x+3, y÷2)

| x | y |
|---|---|
| 0 | 0 |
| 0 | 2 |
| 2 | 2 |
| 2 | 0 |

| x+3 | y÷2 |
|---|---|
| 0+3 | 0÷2 |
| 0+3 | 2÷2 |
| 2+3 | 2÷2 |
| 2+3 | 0÷2 |

| | |
|---|---|
| 3 | 0 |
| 3 | 1 |
| 5 | 1 |
| 5 | 0 |

Following rule A, the square was moved up and a little to the right. In accord with rule B, it was slid sideways to the right and squashed down into a rectangle. Finally, under rule C, it was moved up, to the right, and squeezed into a smaller square, one fourth its original size. Had a triangle been the subject, it could have been turned upside down (an upside down square wouldn't be very interesting), twisted 90° one way or another, and stretched out—it could have been made to dance up and down across the grid, dance from top to bottom and back again.

In all of this activity, pupils are learning the properties of a square, rectangle, and any other shape they are called upon to use. They become familiar with naming the points on a grid or graph, and they begin to feel the power of a formula.

In addition, they are taught to think about size in terms of square units—about area. They learn to talk about linear units—about perimeter. They develop the ability to keep distinct those two kinds of measurement often so fuzzy in the minds of most pupils.

Finally, they have been introduced to a geometry as free of ruler measurement as is possible, and they have seen geometry integrated with the numbers of arithmetic. Therefore, when children do use any scale to measure real objects, they will realize that there are abstract relationships utterly independent of what are commonly called measurements. Such an approach ought to help guide children to reasonable conclusions based on those measurements.

Geometry instruction begins in first grade, building on indistinct ideas brought from Kindergarten and from home and playground. These ideas are continually brought into better focus and expanded into new areas of inquiry. The ground work is gradually laid for a mathematical as well as intuitive understanding of geometry.

## A FINAL WORD

The Math Workshop for Children presents no new mathematics. We have not spun novel theories of education. Rather, we have tried to bring into a unified whole the most universally accepted ideas of both educators and mathematicians.

This general guidebook was not meant to be a comprehensive statement of the ideas, the structure, and the methods of the Math Workshop for Children. Rather, in this book, *we have presented episodes and illustrations of real mathematics to replace meaningless exercises and slogans.*

"Discovery," "individual differences," "understanding," are labels that every mathematics program uses. We do too, but we have tried to define these terms by examples.

We have illustrated activities which provide an environment for discovery. We have indicated a score or more of open-end experiences to challenge the most gifted children and to provide a solid feeling of success for slow children. We have outlined new approaches in many areas in which teachers have had to resort to rules because "understanding" seemed so elusive.

One of the ways in which we have attempted to enrich the study of mathematics is by employing a host of interesting and intriguing devices. For example, the cross number puzzle fascinates children and adults. There is some fascinating meaning hidden in this versatile form. What is it? The search for this meaning directs the pupils' attention to the fundamental laws of arithmetic. There is no trickery or accident involved here—only rock bottom mathematics. No wonder this little device stirs excitement every time it is let loose in the classroom.

In the Math Workshop for Children desires are created to "fill in the table"—to "see what's going on." A half page of computations may be meaningless by itself, but recorded in an orderly fashion on a chart, the figures may reveal a mathematical structure that sparkles and invites further exploration.

When children and adults see 100 or 150 numbers arranged in a page-sized grid for study, for coloring in patterns, or as an arena for jumping back and forth to add and subtract, then the familiar number system itself becomes a playground, a laboratory, the terrain for a treasure hunt.

The first grade materials are exciting introductions to every mathematical idea: to counting, to addition, to multiplication, to geometry, to graphs, to discussions and arguments that lead to hunches, and to proofs.

In short, we have tried in this book to demonstrate by examples that mathematical ideas can be suggested to young minds in ways that capture interest. We have detailed some of these concepts as they unfold, layer by layer, grade by grade, into mathematical understandings more sophisticated than might have been imagined possible. This is what we mean by the "spiral curriculum;" ideas are introduced early in the child's experience and are examined over and over again from new viewpoints. As the child moves up from grade to grade, the foundations of his understanding of mathematics will be strengthened, and his insights will expand on every horizon.

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